

# We've now seen...

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- Building of several sheaf models
- Inferring/imputing missing or noisy data using the sheaf
- But what of the domain of validity?

## Why sheaves?

Sheaves:

- Are the **universal reductionist paradigm** that guide the composition of more complicated models from simpler ones
- Moderate between **different levels of abstraction** and/or domains of validity for models

And recently, they can handle noisy real-world data with practical models in software



Michael Robinson



Reference: <https://doi.org/10.32408/compositionality-2-2>

Michael Robinson

# What's the right domain of validity?

- How many variables do you really need?
- Concrete example: counting stars in a star cluster

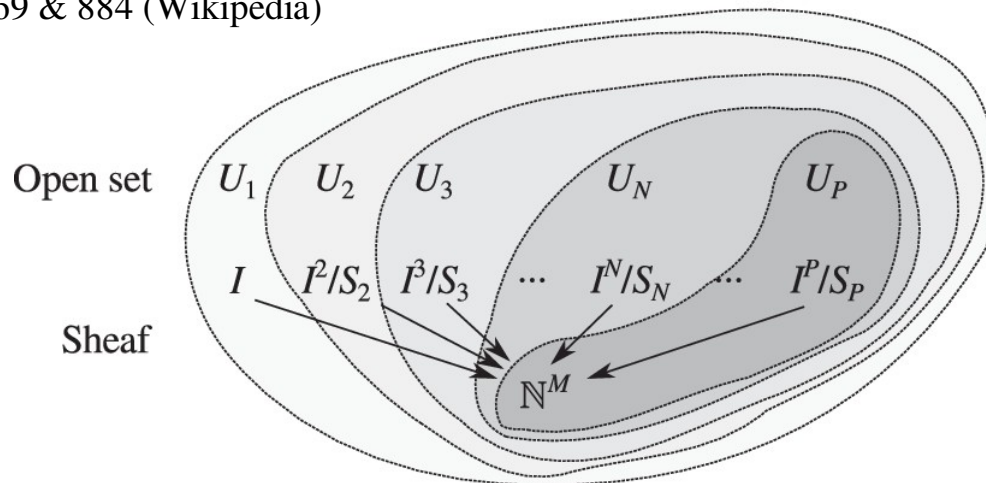


NGC 869 & 884 (Wikipedia)

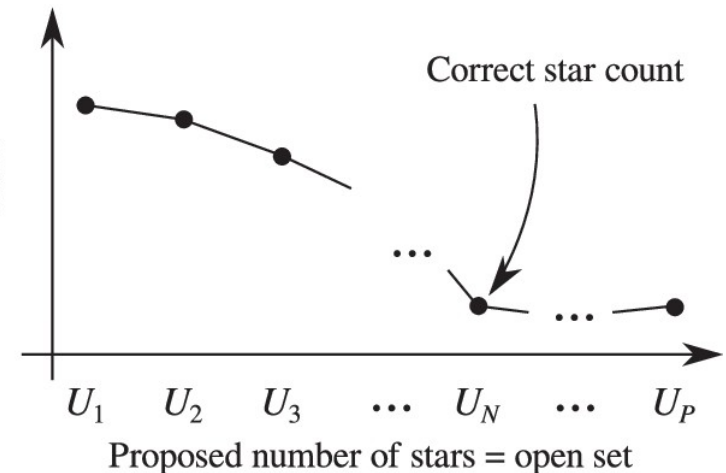
Position and brightness =  $I$

Variables for  $P$  stars:  $I^P/S_P$

"Symmetric group on  $P$  elements" = ignore order

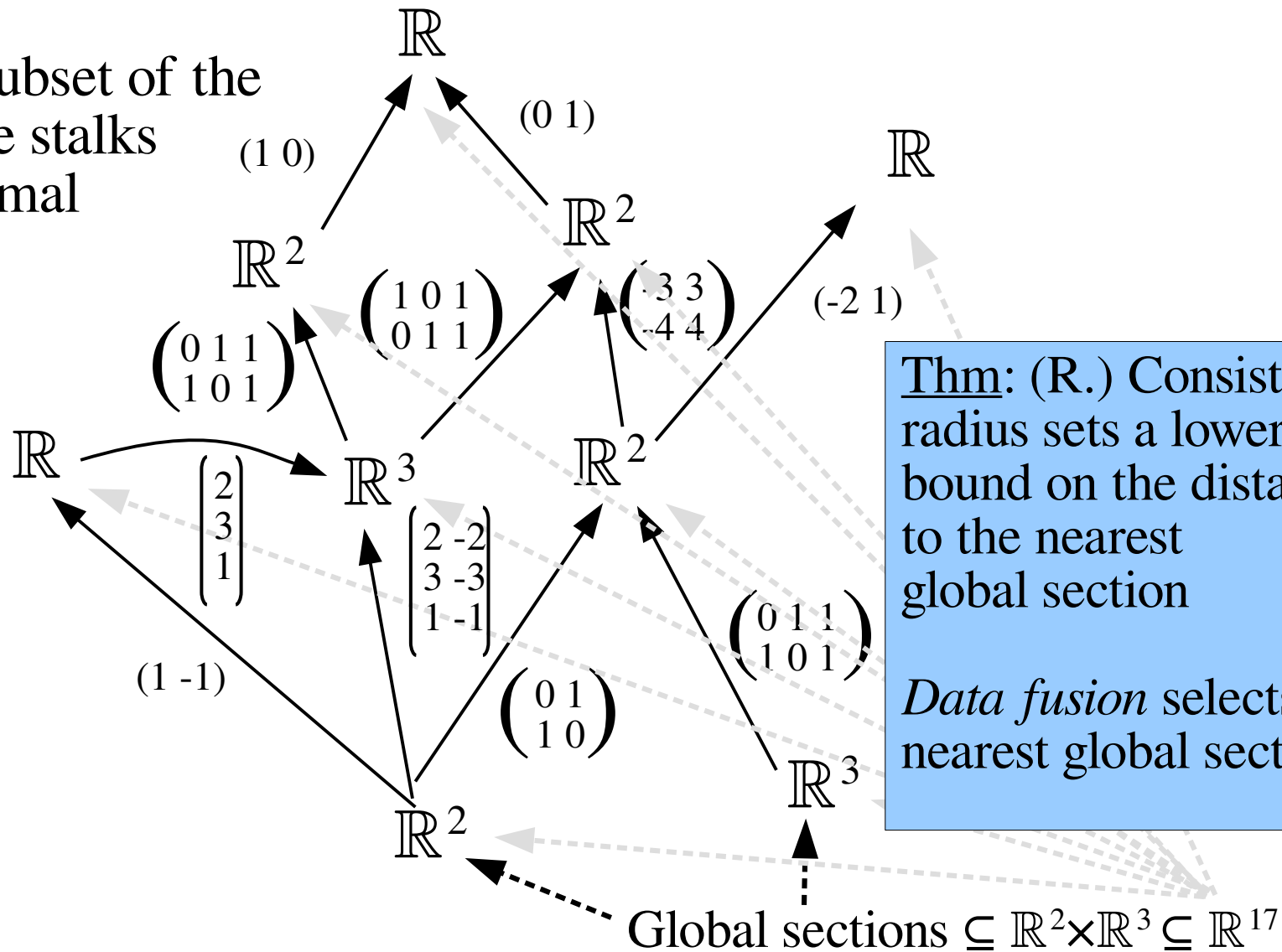


local consistency radius



# The space of global sections

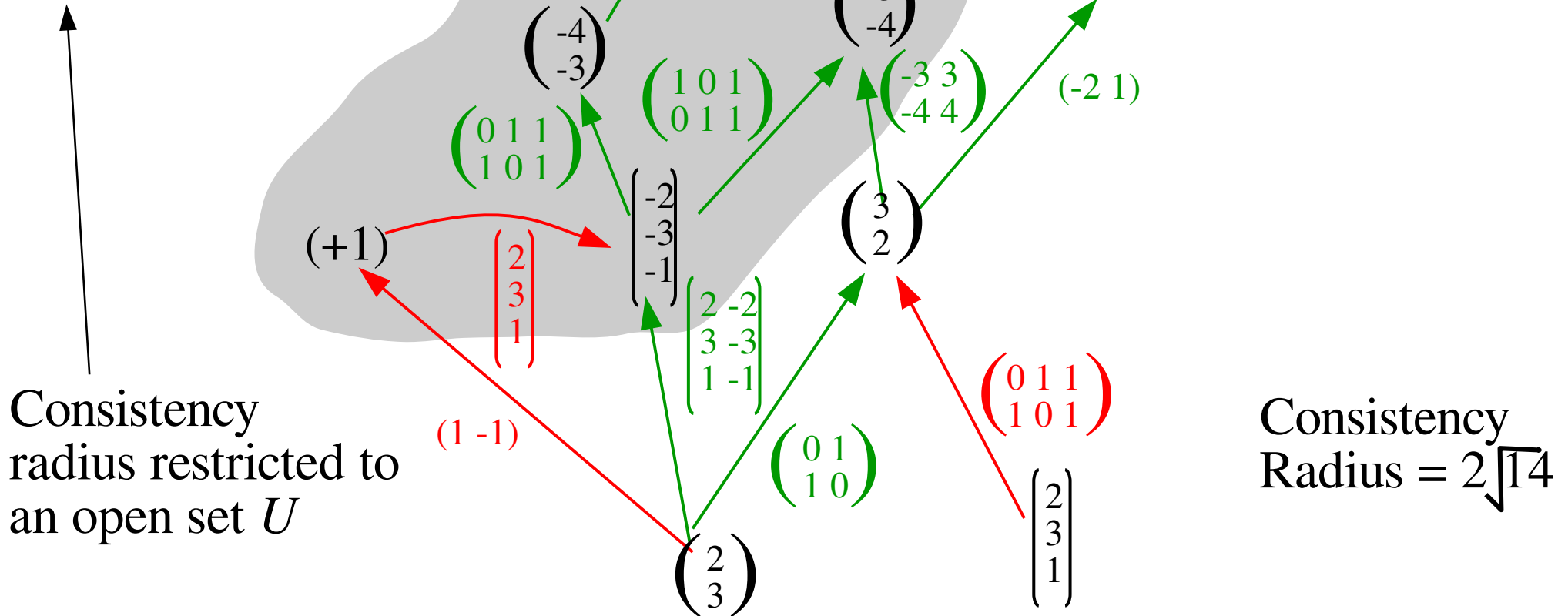
It's a closed subset of the product of the stalks over the minimal elements





# Consistency radius is monotonic

Proposition:  
If  $U \subseteq V$  then  
 $c(U) \leq c(V)$



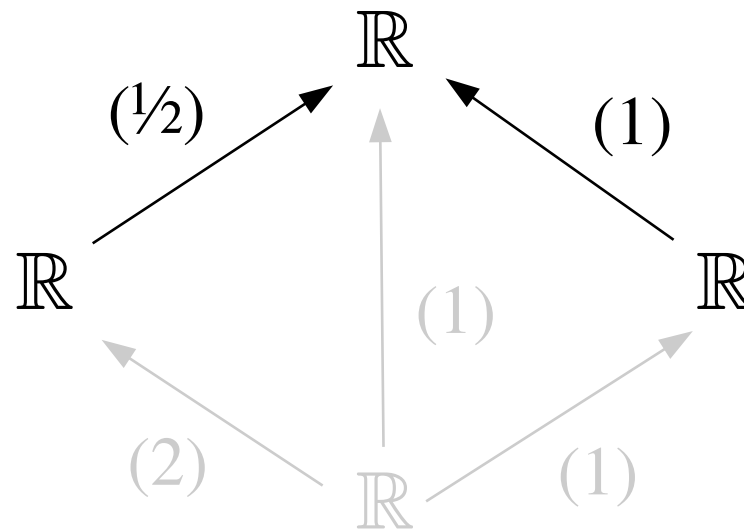
Note: lots more restrictions to check!



# Consistency radius optimization

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NB: restrictions act by multiplication

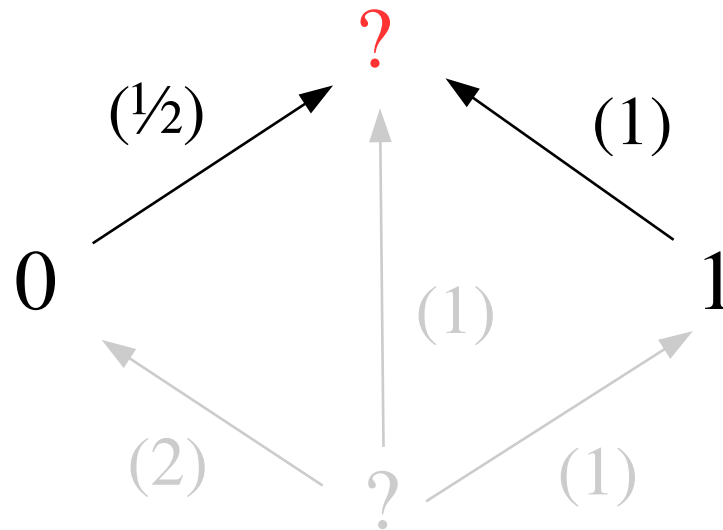


This is a sheaf on a small poset



# *Consistency radius optimization*

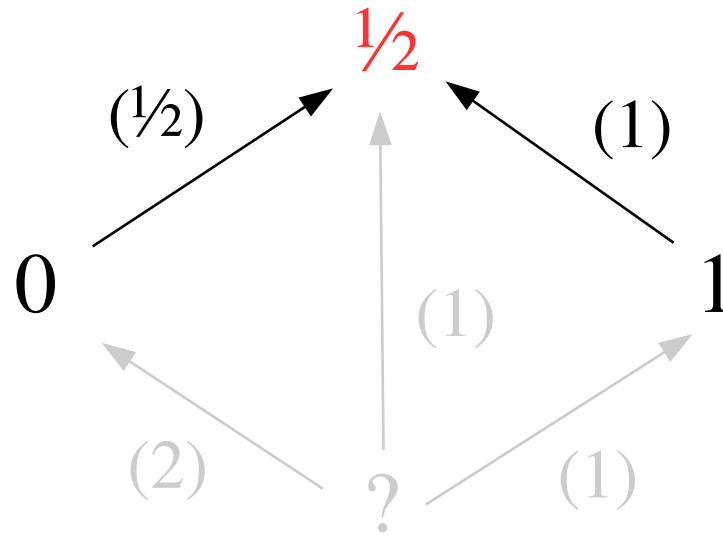
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Here is an assignment supported on part of it

# *Consistency radius optimization*

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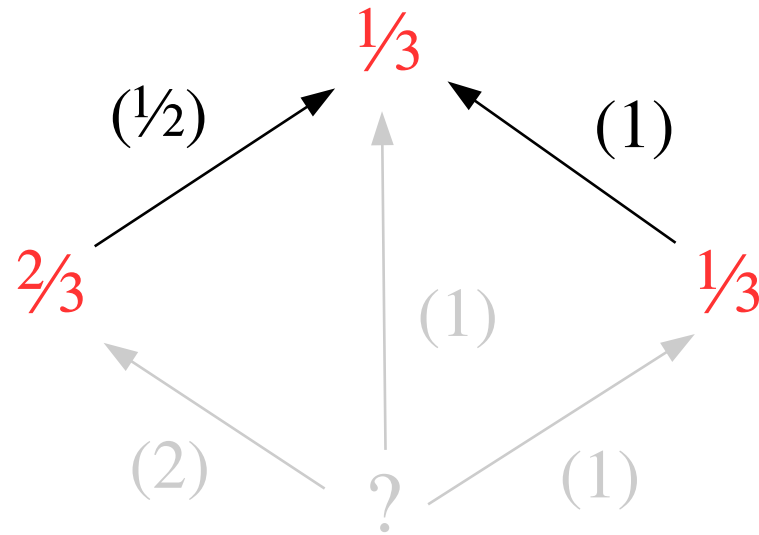
Minimizing the consistency radius when extending globally





# Consistency radius optimization

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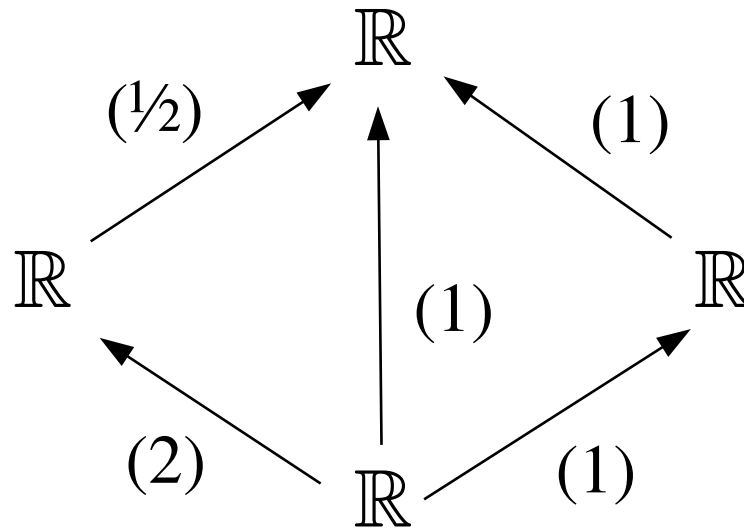


Here is the closest global section (everything can be changed)



# Extending to a larger sheaf...

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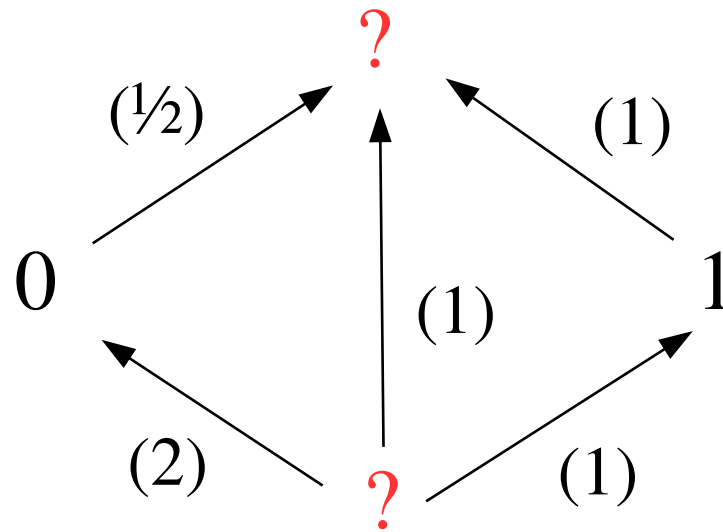


This is the full sheaf diagram including all open sets in the Alexandrov topology, not just the base



# Extending to a larger sheaf...

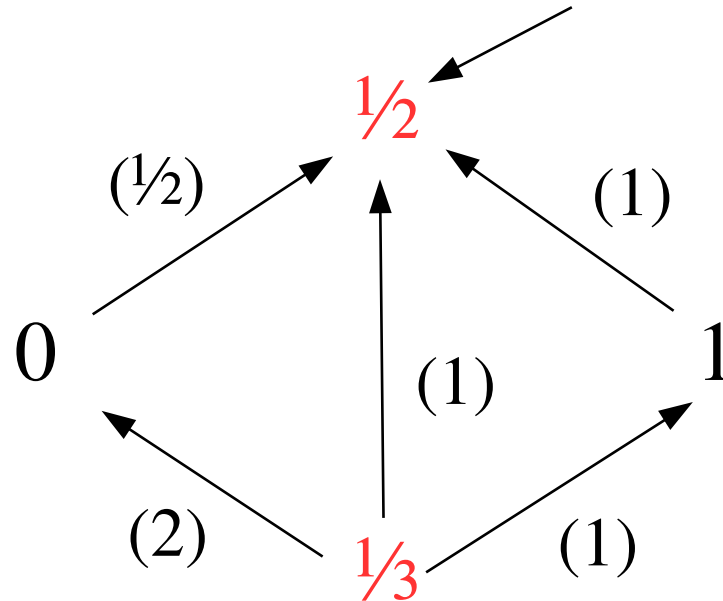
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# Extending to a larger sheaf...

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This value can be anything between  $\frac{1}{3}$  and  $\frac{2}{3}$

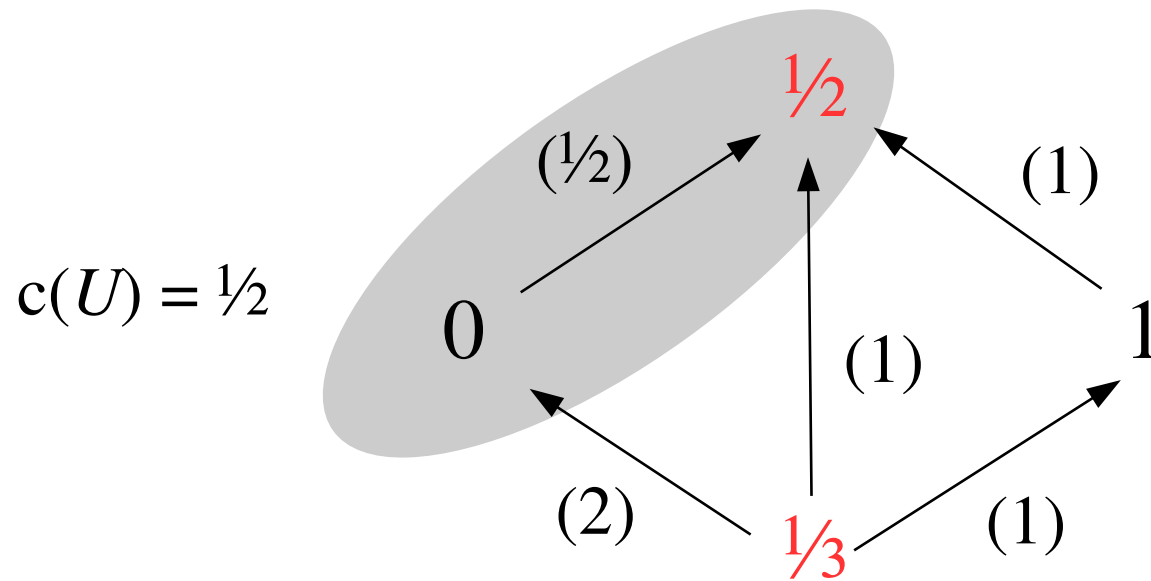


Minimizing the consistency radius when extending  
The value on the intersection is no longer unique!



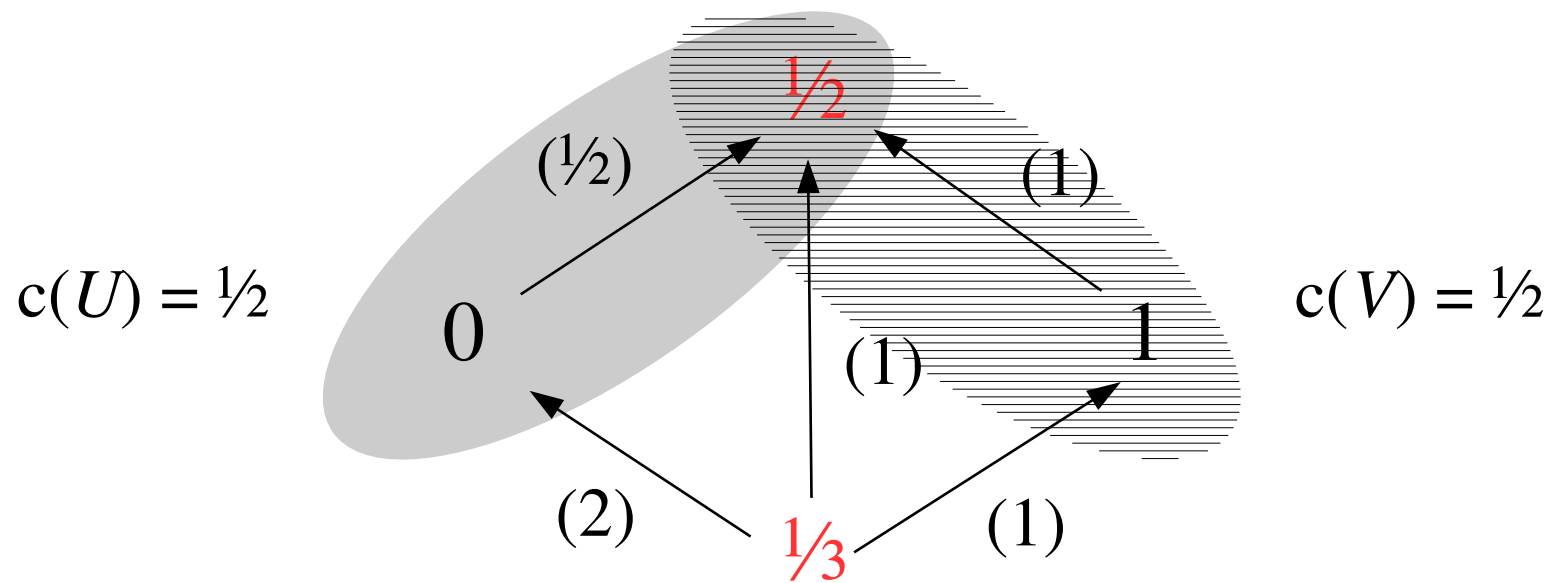
# Extending to a larger sheaf...

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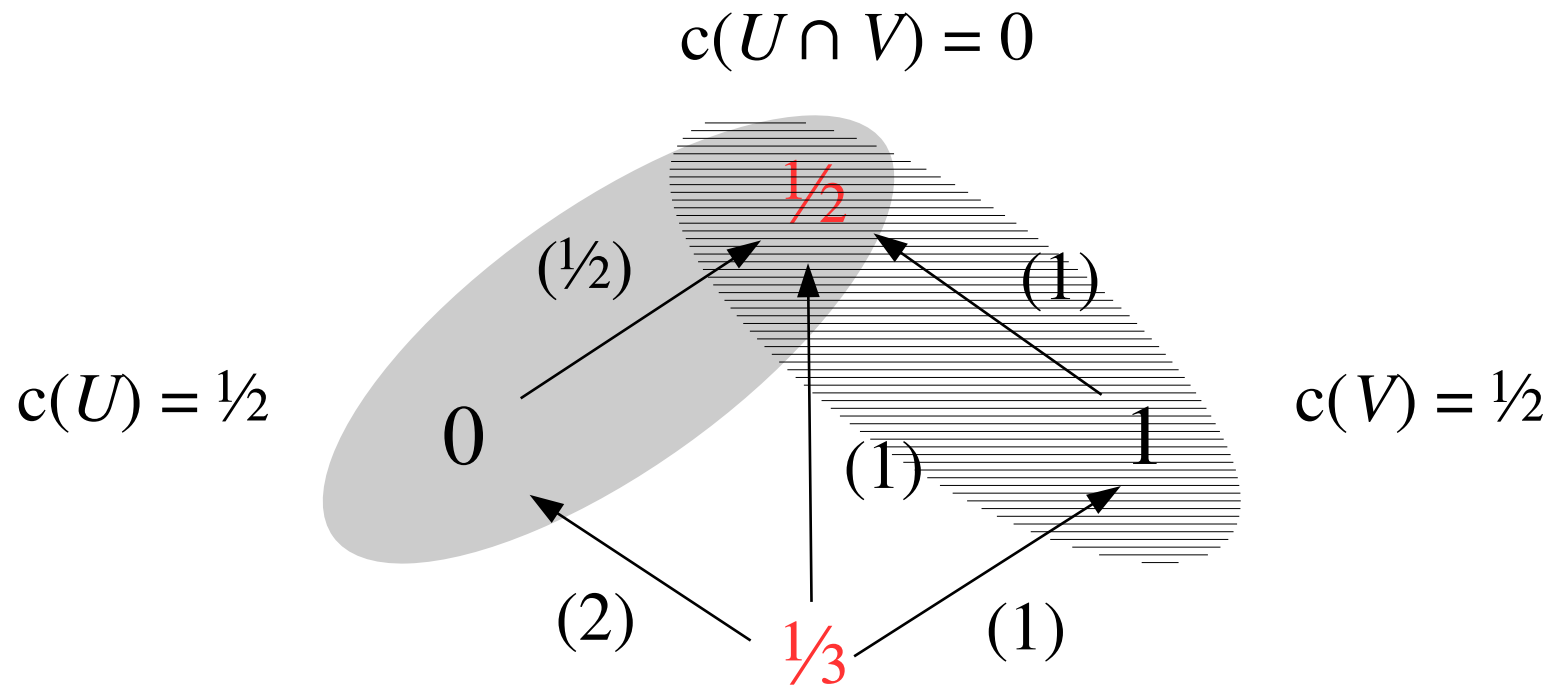
# Extending to a larger sheaf...

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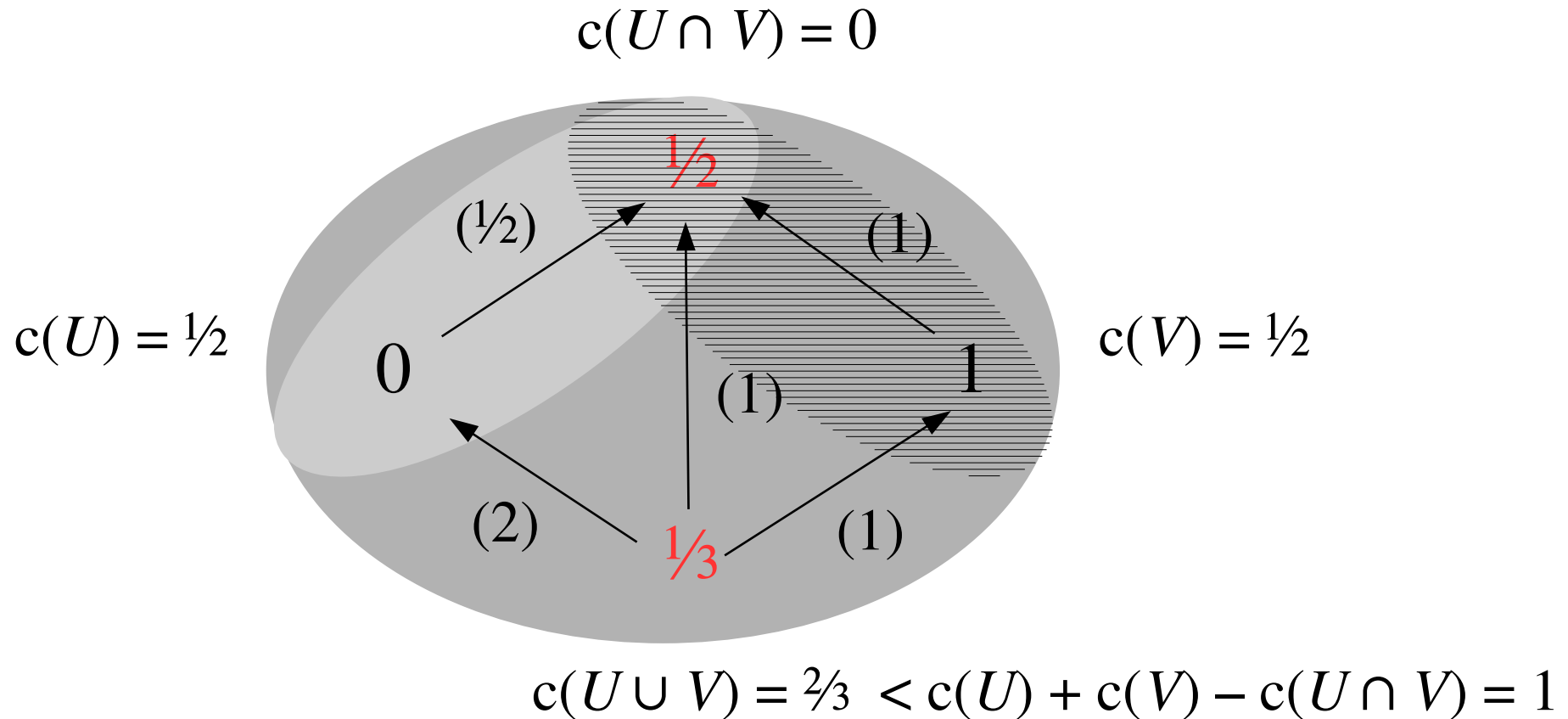


# Extending to a larger sheaf...

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# Consistency radius is not a measure



Proposition:

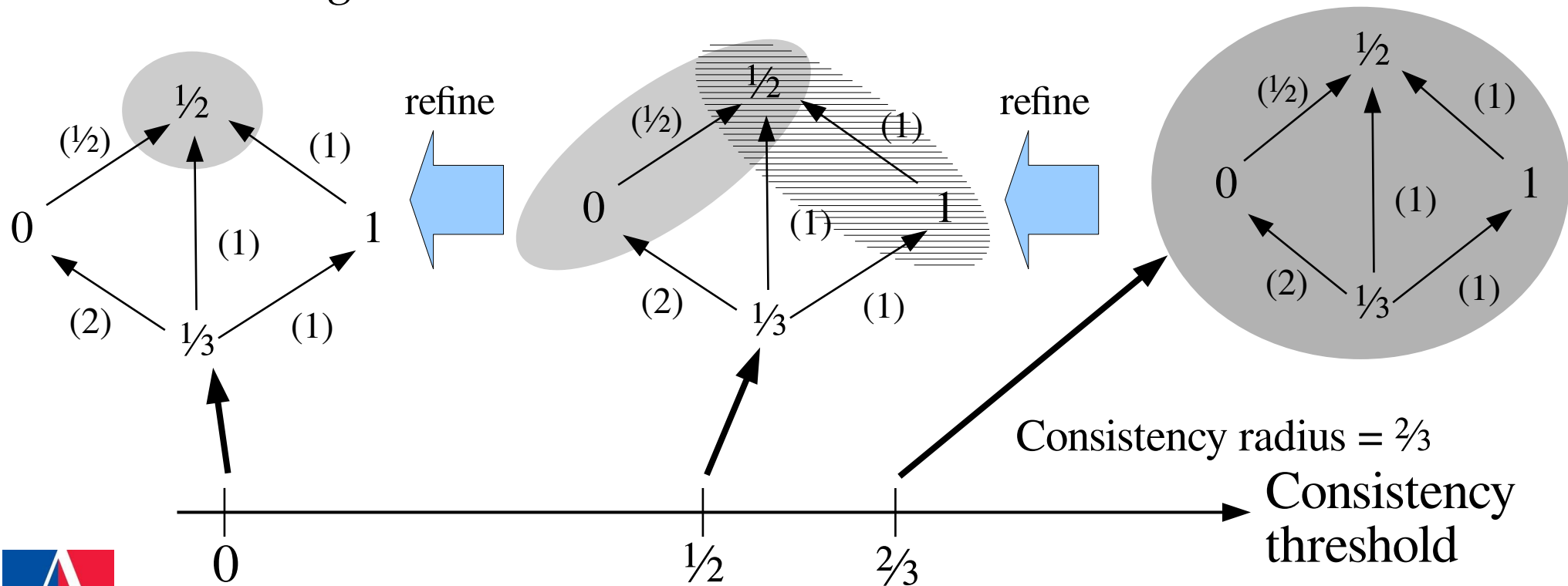
“Local consistency of a Global assignment” is a (loose) upper bound for “Global consistency of a Local assignment”





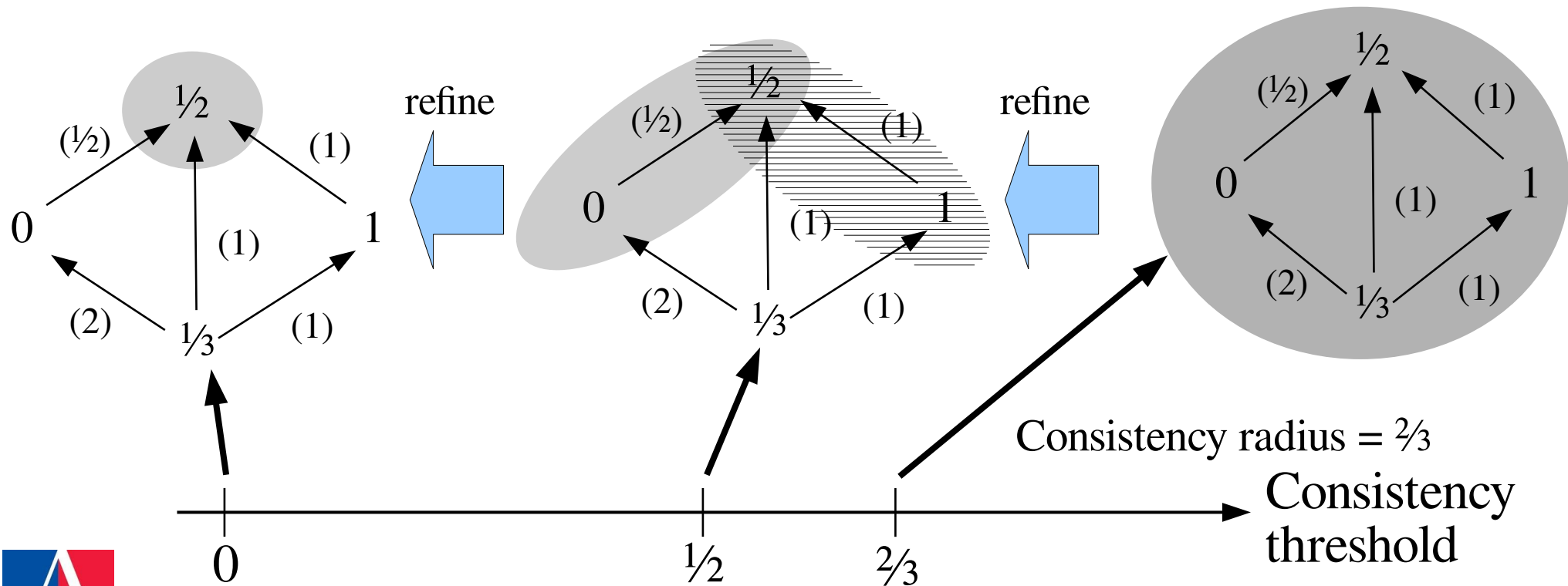
# The *consistency filtration*

- ... assigns the set of open sets (open cover) with consistency less than a given threshold
- **Lemma:** consistency filtration **is itself a sheaf** of collections of open sets on  $(\mathbb{R}, \leq)$ . Restrictions in this sheaf are *cover coarsenings*.



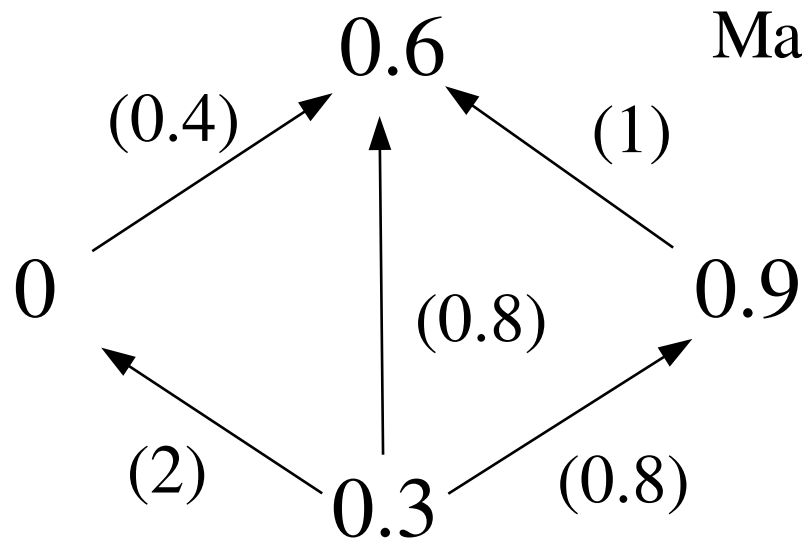
# Consistency filtration is natural

- Theorem: Consistency filtration is continuous under the an *interleaving distance*
- Theorem: Consistency filtration is also functorial
- (Note: the proof is quite intricate...)

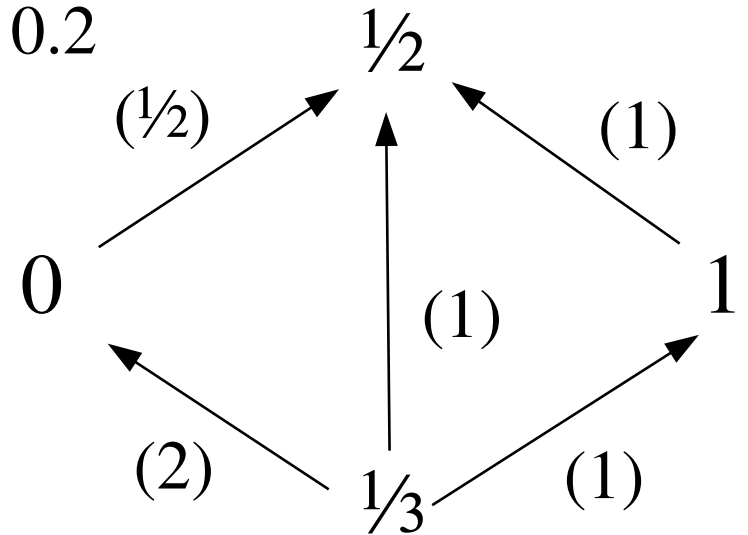


# A small perturbation ...

- Perturbations allowed in both assignment **and** sheaf (subject to it staying a sheaf!)

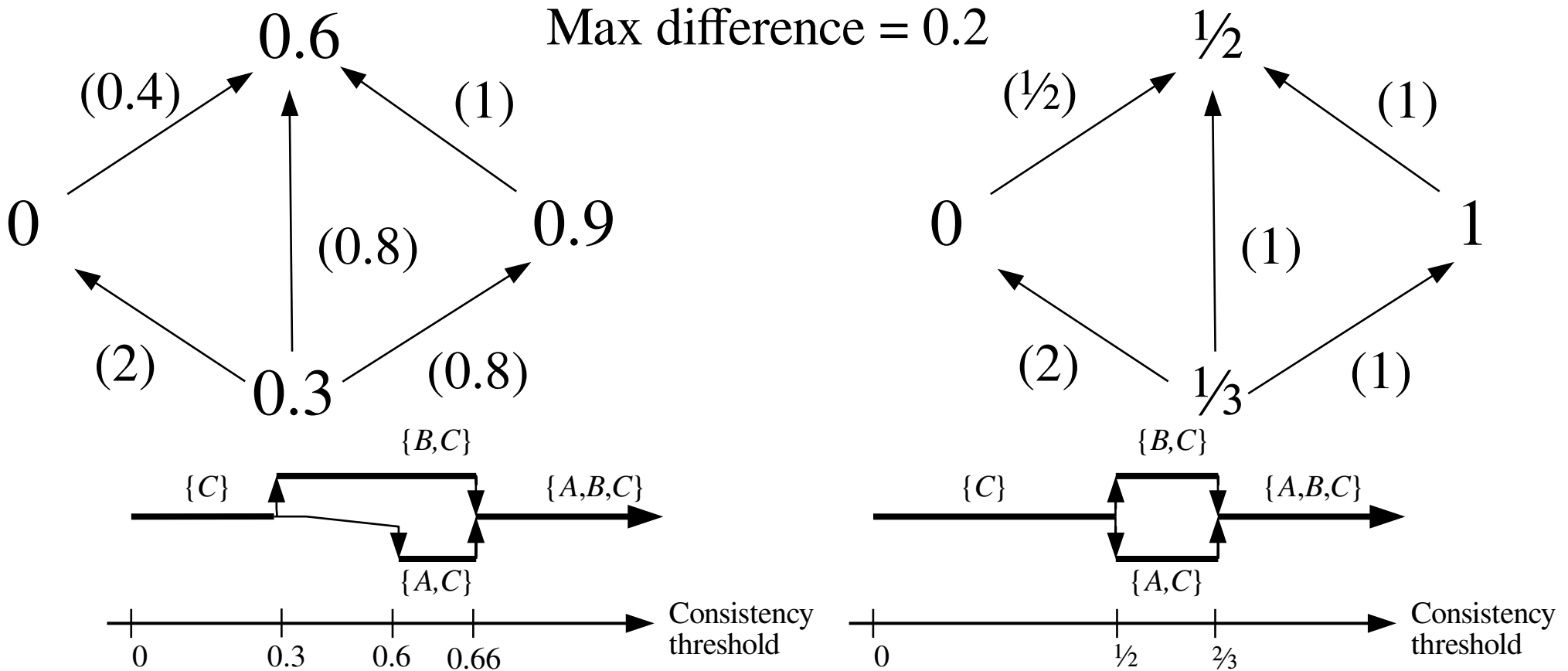


Max difference = 0.2



# A small perturbation ...

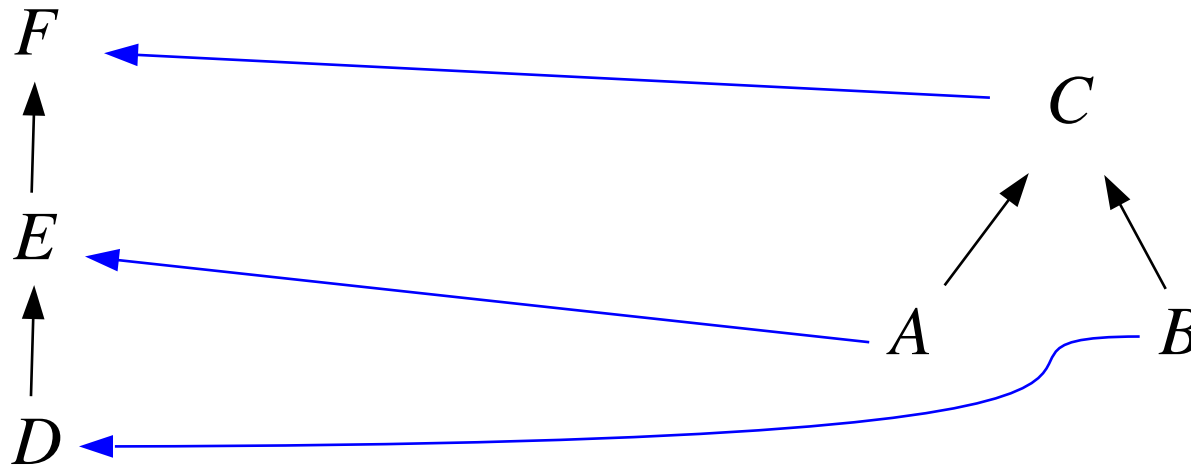
- Compute consistency filtrations... they're similar



# A sheaf assignment *morphism* is ...

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- ... first, an order preserving map between base posets...

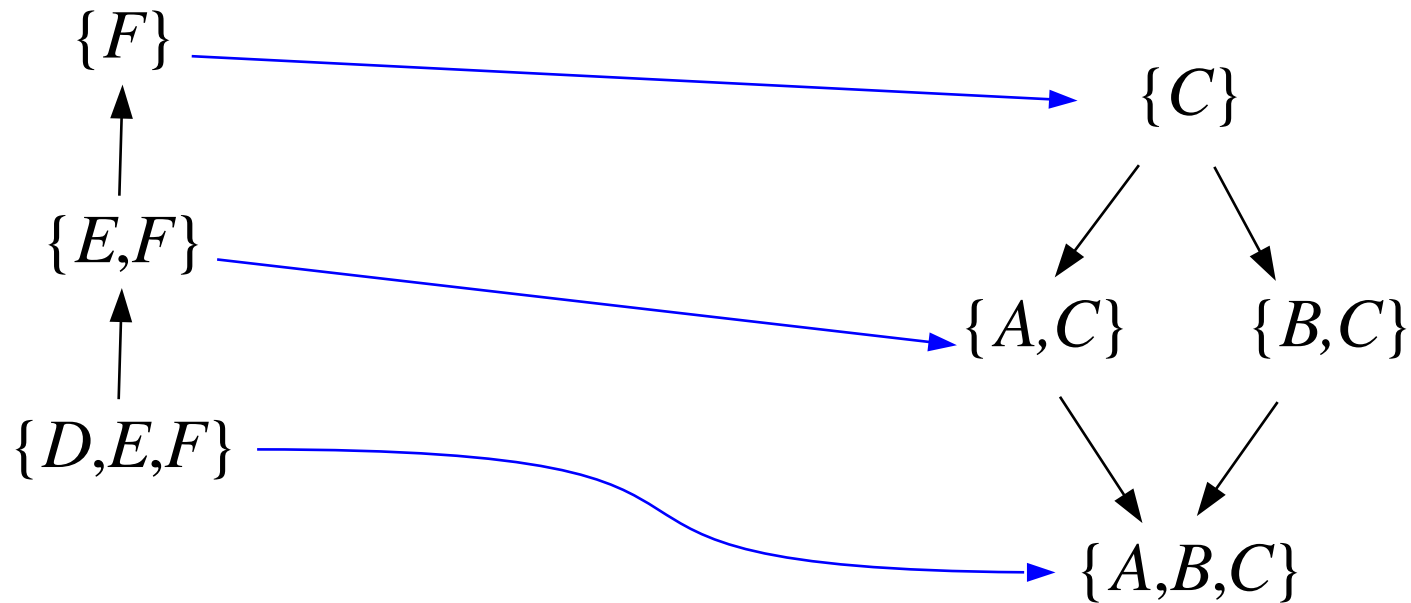


# A sheaf assignment *morphism* is ...

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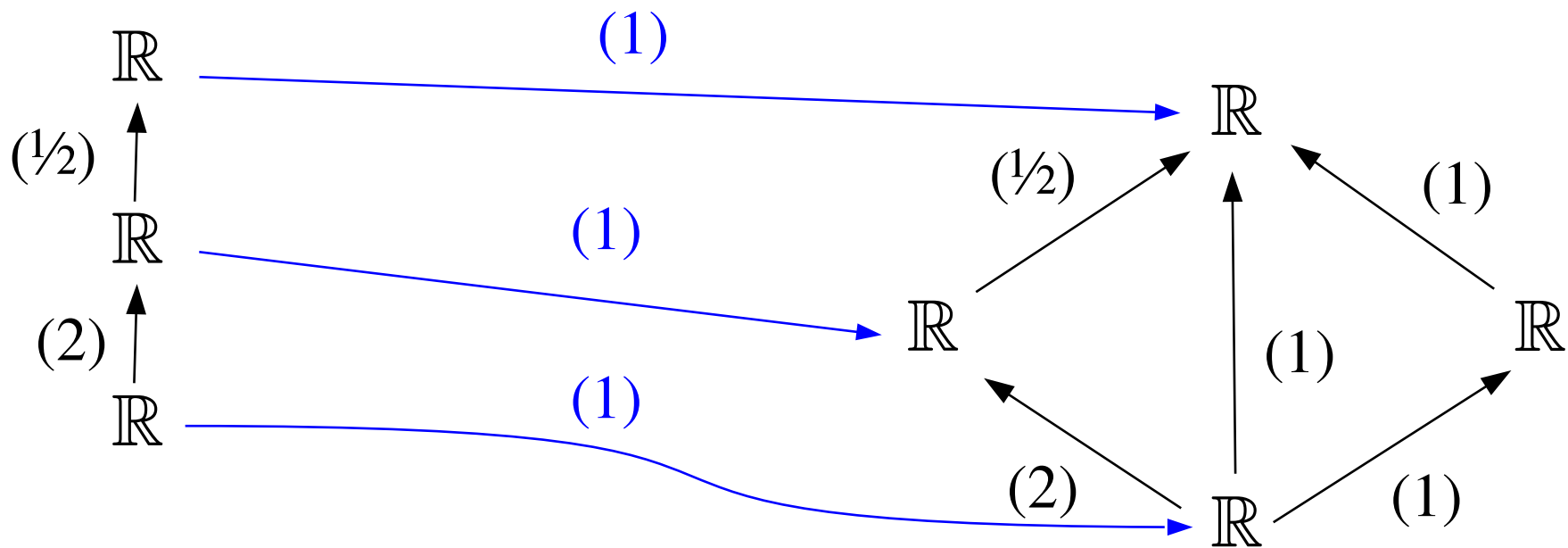
- ... which is a continuous map ...

(preimages shown below)



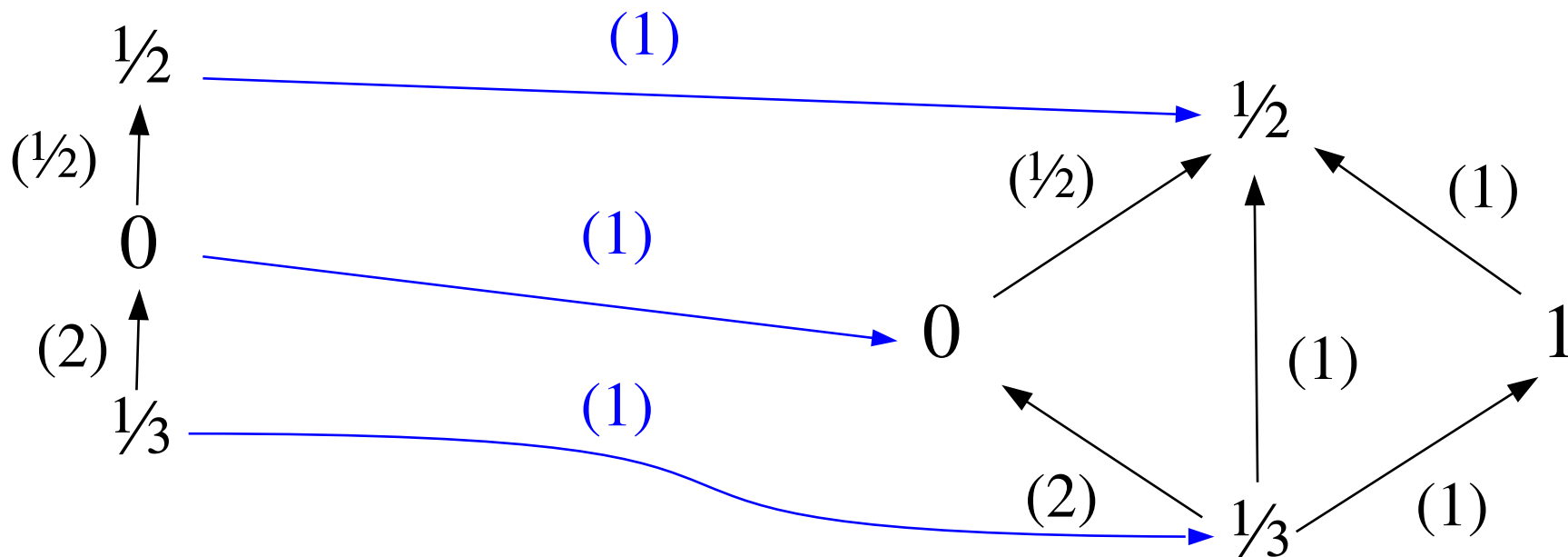
# A sheaf assignment *morphism* is ...

- ... add to this, a commuting set of *component maps* for the two sheaves ...



# A sheaf assignment *morphism* is ...

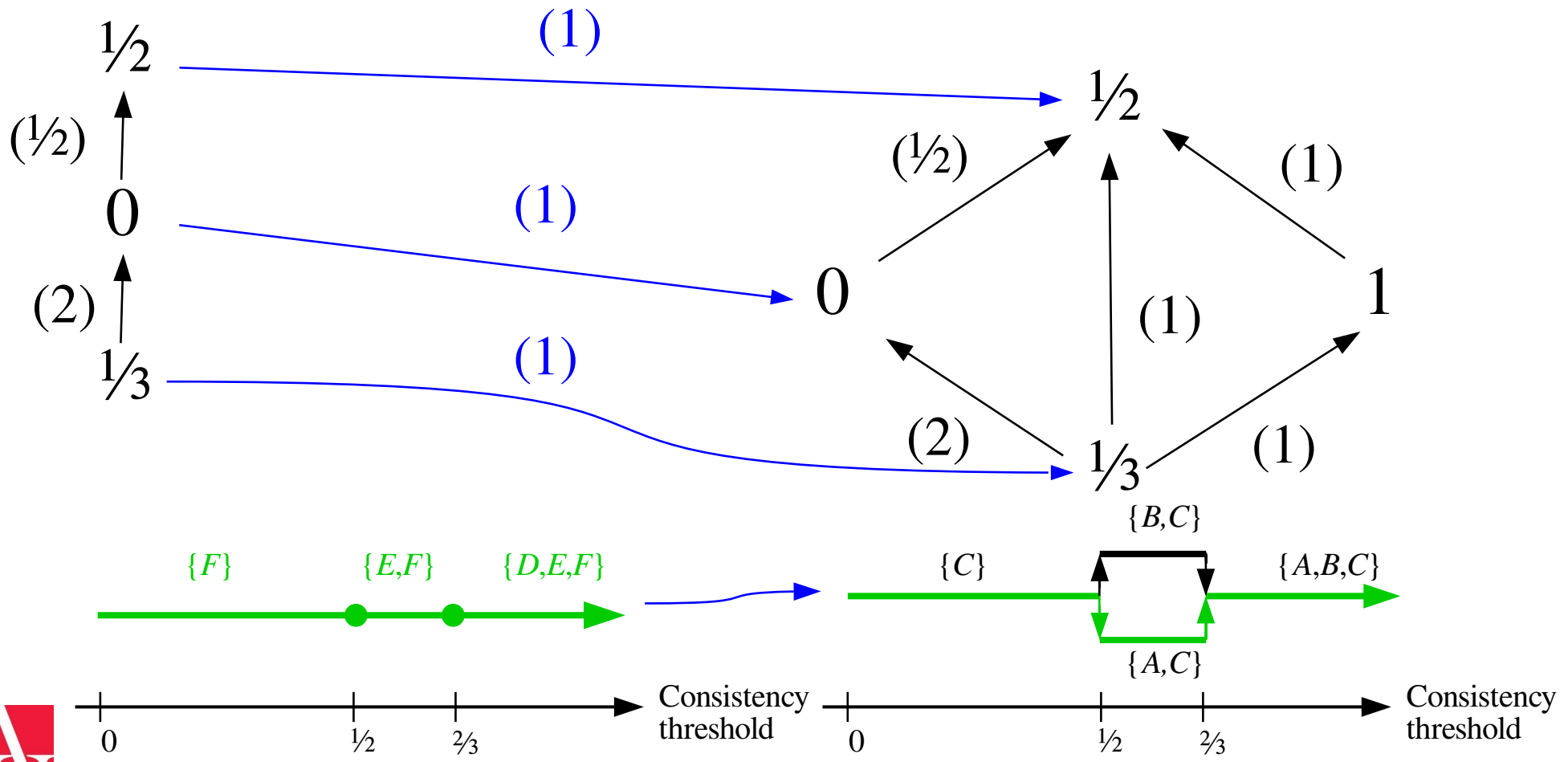
- ... such that the assignments on both ends are preserved.





# Functoriality!

- Compute consistency filtrations, and all that's really needed is to align the open sets in the covers!



# Interpretation

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- Sheaf: a **data structure** for modeling consistency
- Assignment: an **instance** of the data housed in a sheaf
- Consistency radius: **how well** do data and model agree?
- Consistency radius optimization: **predict** some missing or cleaner data
- Consistency filtration: **where** do data and model agree?

