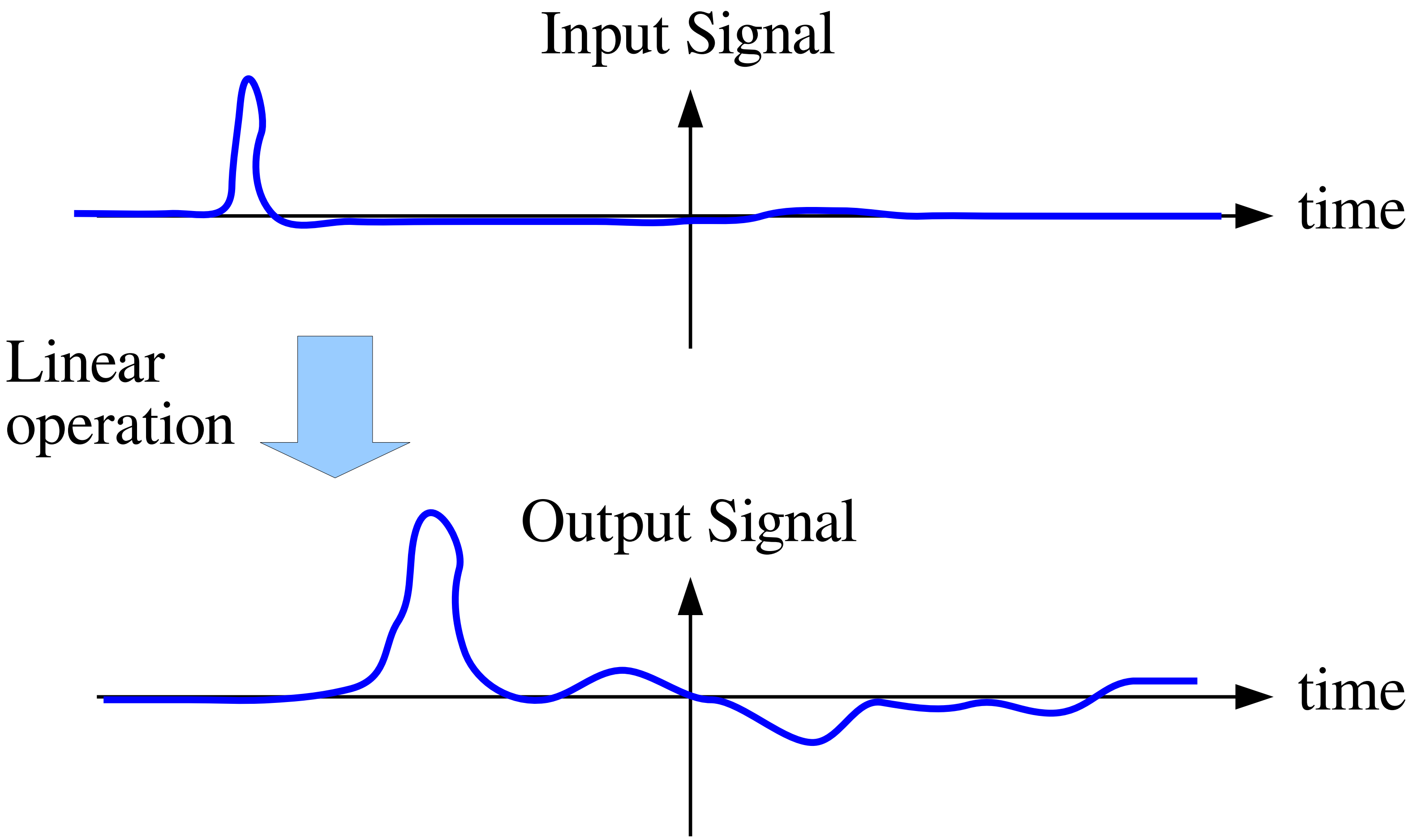
We've now seen...

- Building of several sheaf models
- Inferring missing/noisy data

- Now how about some different data? Timeseries!

Discrete-time LTI filters

- ***Linear*** *Translation-Invariant* filters are the workhorses of modern signal processing



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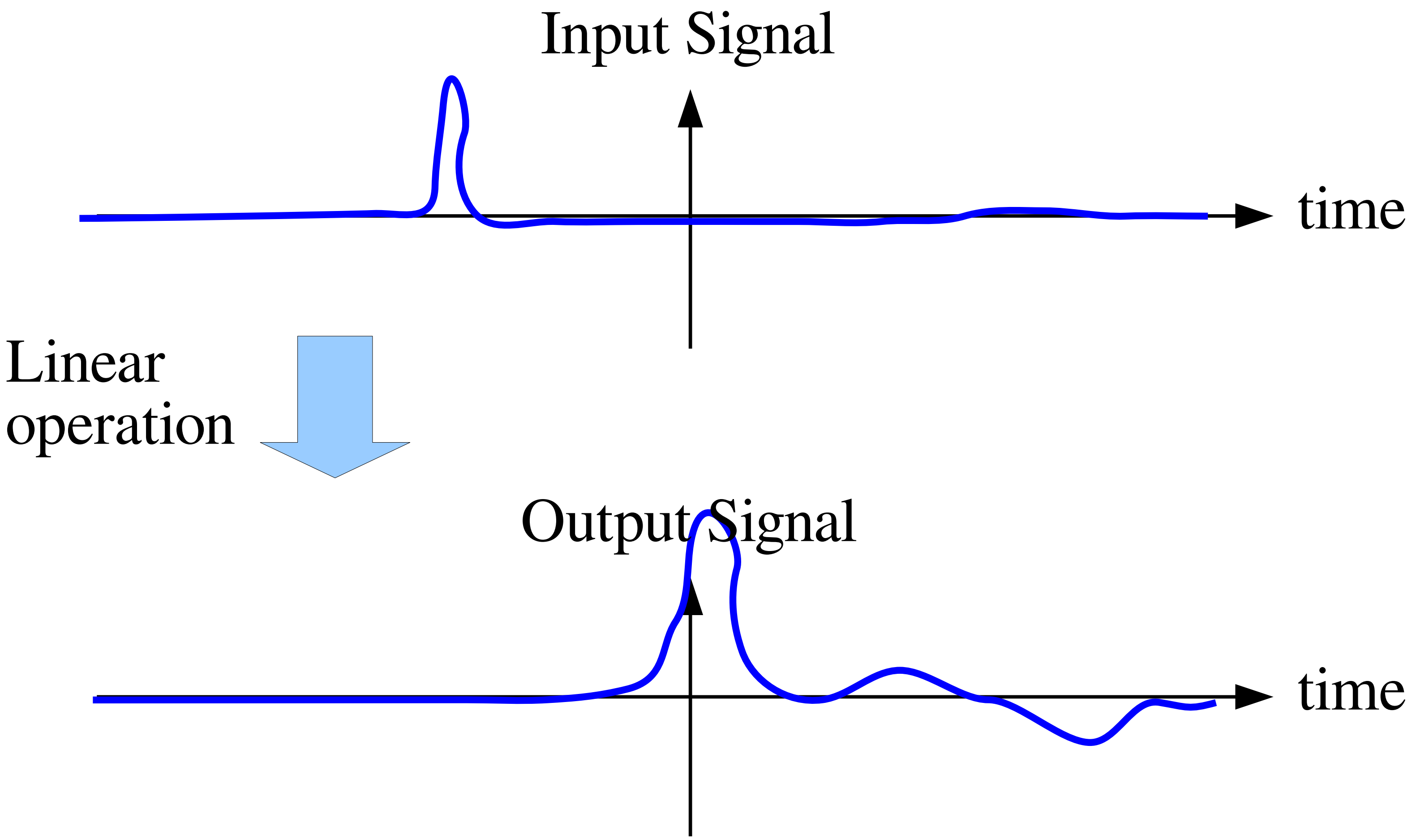

Discrete-time LTI filters

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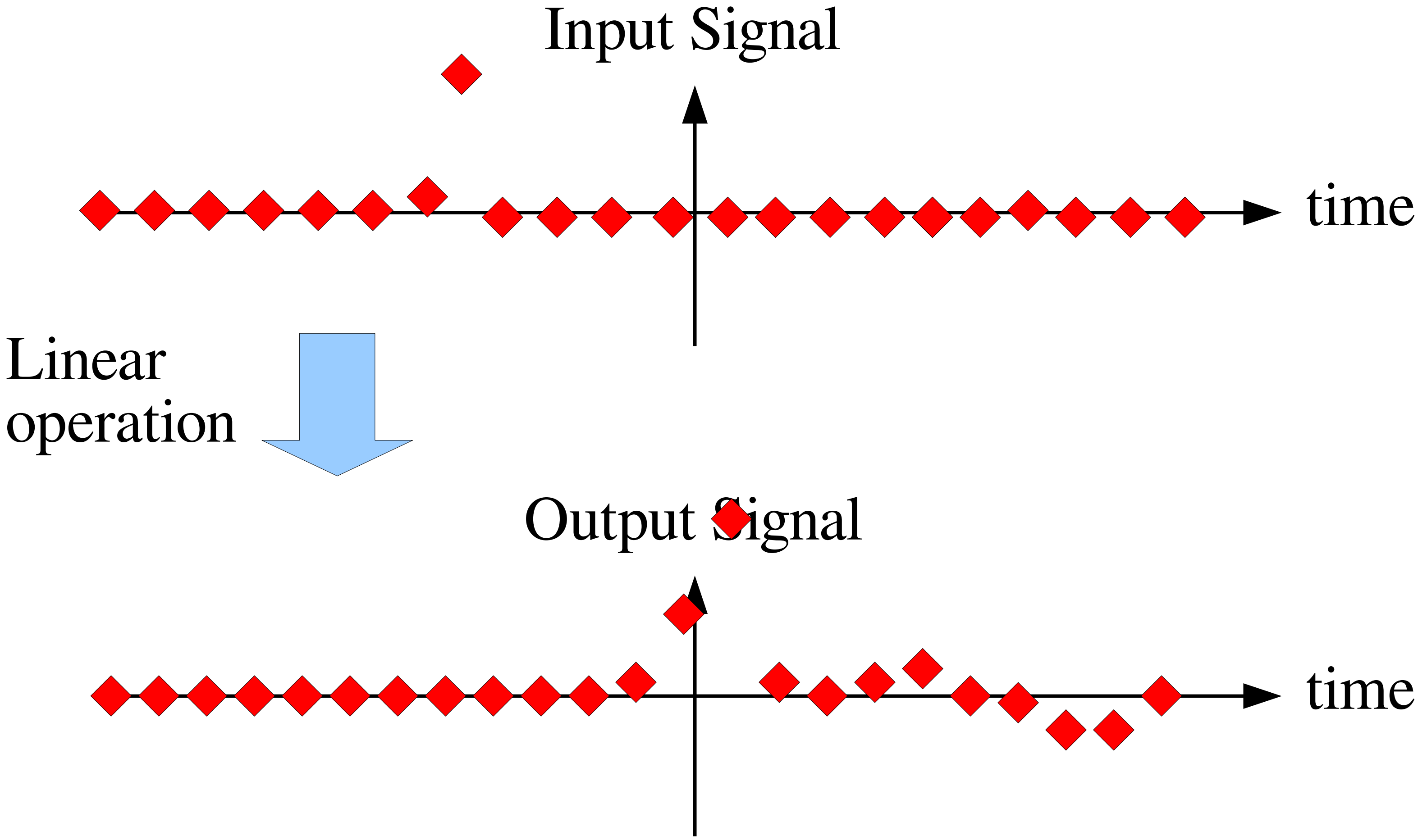


Filters as sheaf morphisms

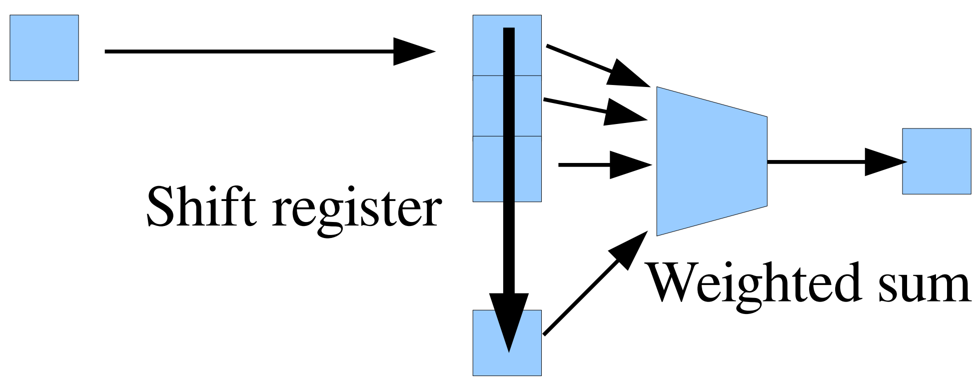
- Theorem: Every discrete-time LTI filter can be encoded as a sequence of two *sheaf morphisms**

$$S_1 \xleftarrow{\text{projection}} S_2 \xrightarrow{\text{combination}} S_3$$

Sheaf formalism

Input — Internal state — Output

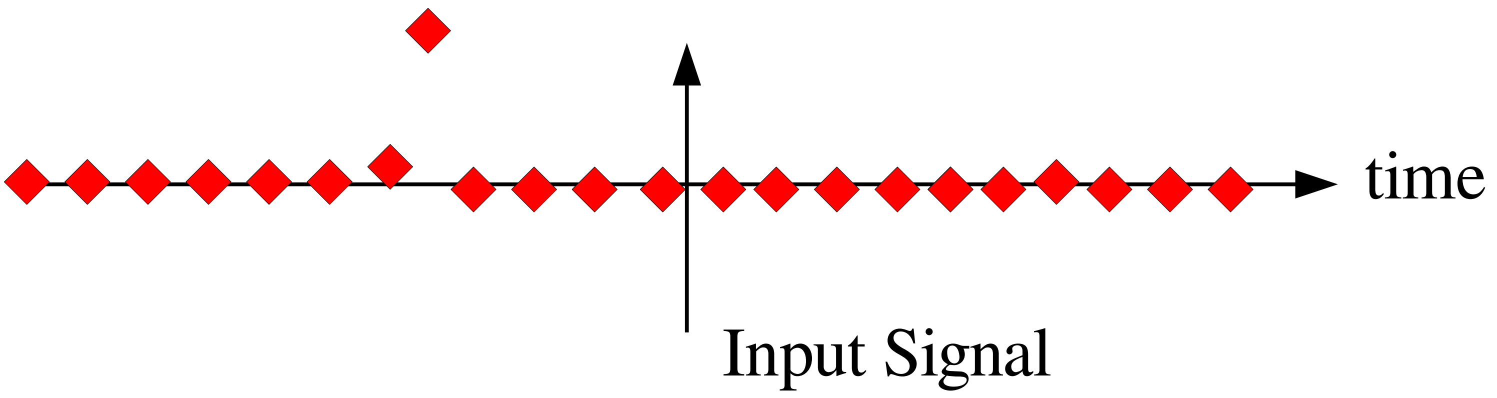
Hardware



*A commutative diagram of maps between the stalks of two sheaves

Proof sketch: Input sheaf

- Sections of this sheaf are timeseries, instead of continuous functions



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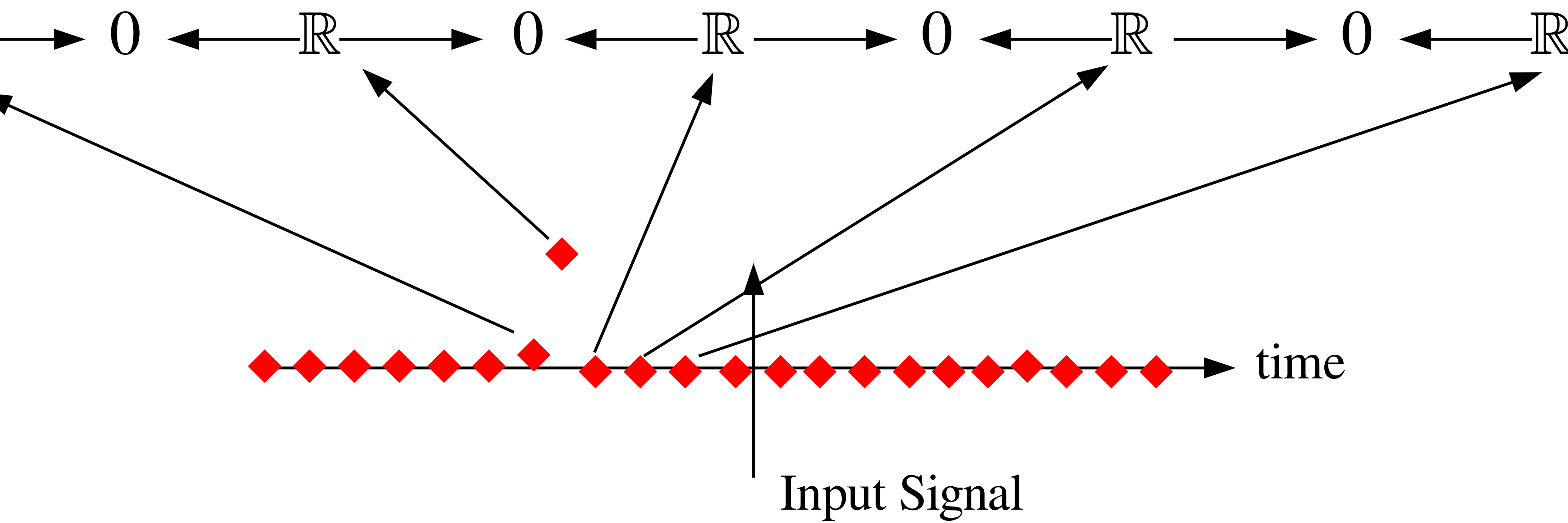


Proof sketch: Input sheaf

- Sections of this sheaf are timeseries, instead of continuous functions

$$\rightarrow 0 \leftarrow \mathbb{R} \rightarrow 0 \leftarrow \mathbb{R} \rightarrow 0 \leftarrow \mathbb{R} \rightarrow 0 \leftarrow \mathbb{R}$$

Proof sketch: Output sheaf

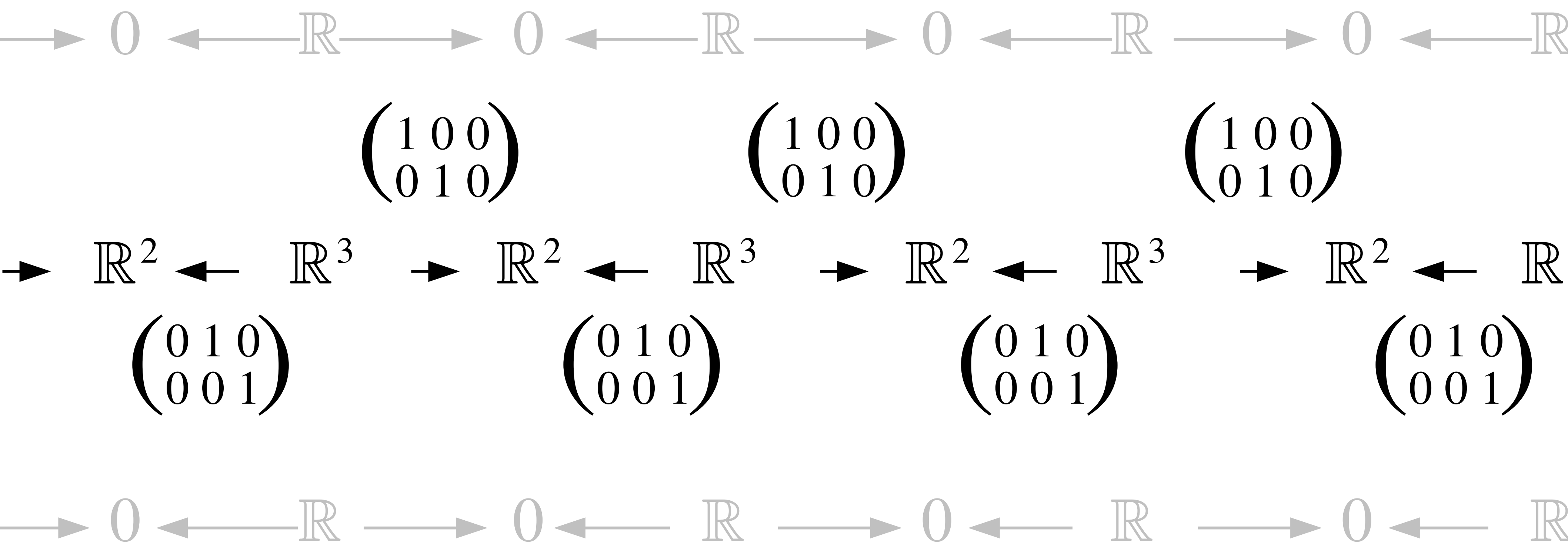
- The output sheaf is the same

$$\longrightarrow 0 \longleftarrow \mathbb{R} \longrightarrow 0 \longleftarrow \mathbb{R} \longrightarrow 0 \longleftarrow \mathbb{R} \longrightarrow 0 \longleftarrow \mathbb{R}$$

$$\longrightarrow 0 \longleftarrow \mathbb{R} \longrightarrow 0 \longleftarrow \mathbb{R} \longrightarrow 0 \longleftarrow \mathbb{R} \longrightarrow 0 \longleftarrow \mathbb{R}$$

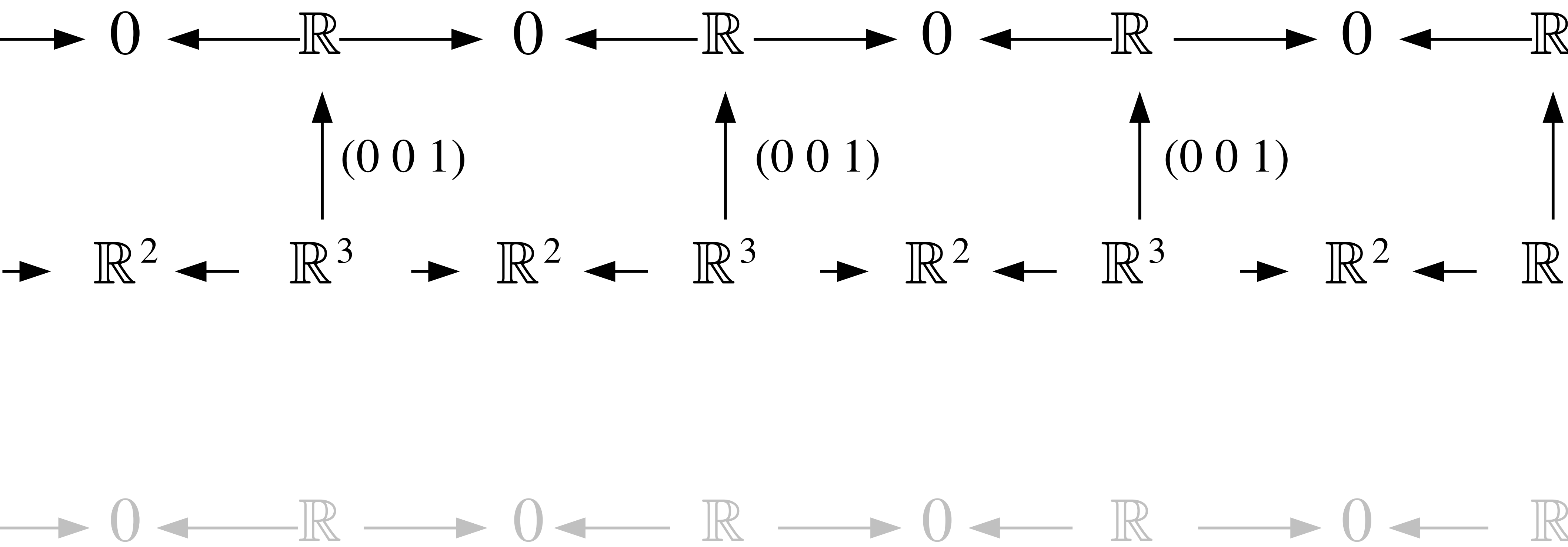
Proof sketch: The internal state

- Contents of the shift register at each timestep
- $N = 3$ shown



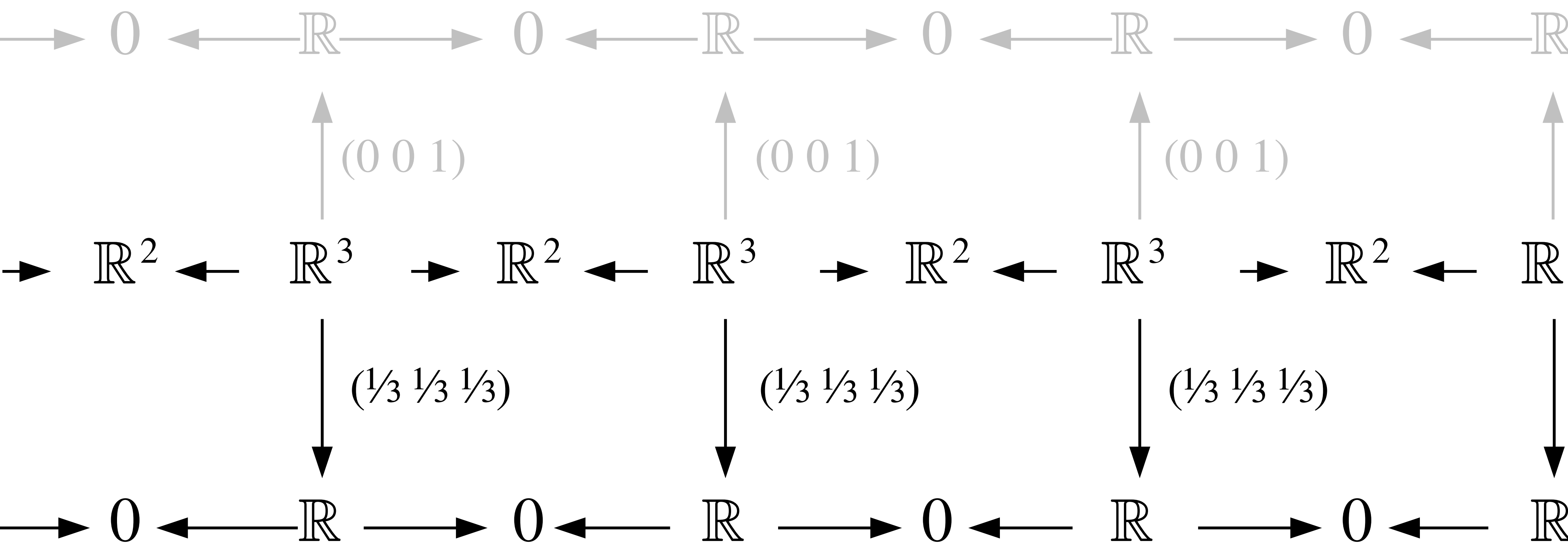
Proof sketch: The internal state

- Loads a new value with each timestep



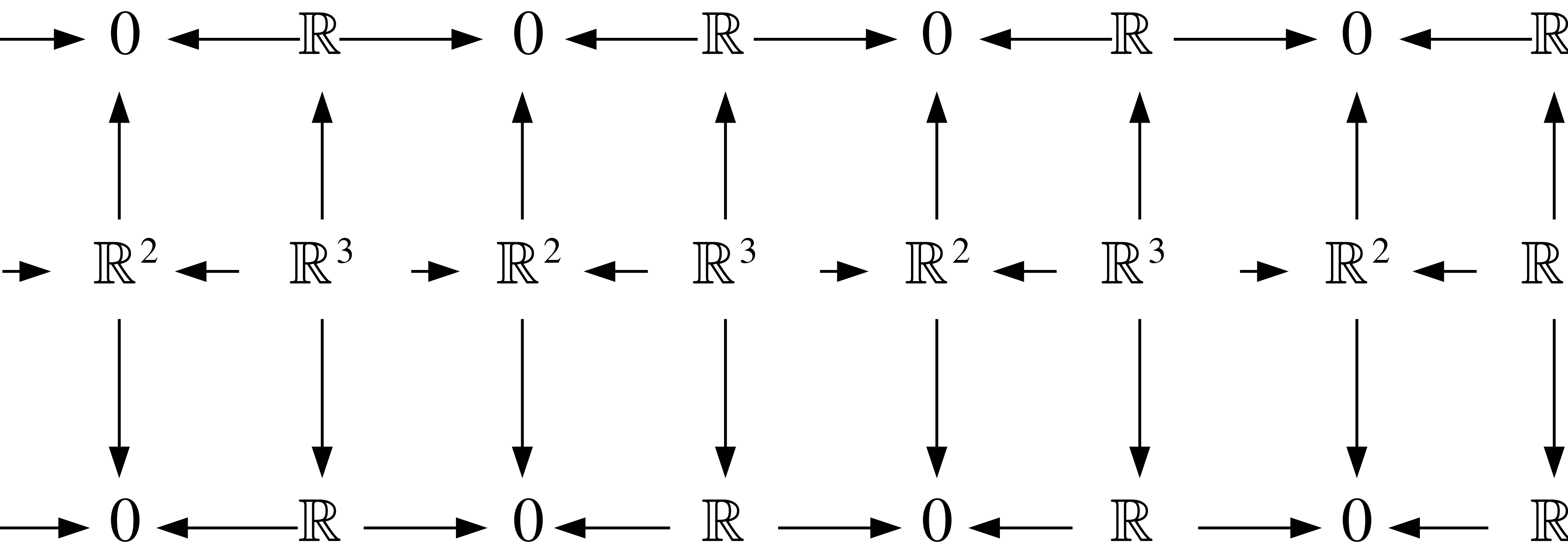
Proof sketch: The internal state

- Computes linear functional of the shift register at each timestep (for instance, compute the mean)



Proof sketch: Finishing both morphisms

- Put in a few zero maps!
- Note that the whole diagram commutes!



Controlled systems

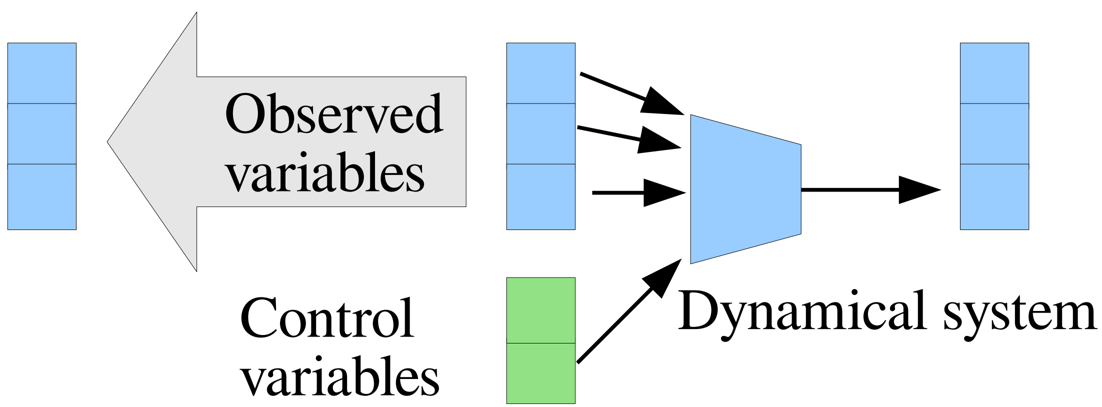
- Theorem: Every discrete-time controlled dynamical system can be encoded as a sheaf diagram

$$\xrightarrow{\text{evolution}} S_1 \xleftarrow{\text{projection}} S_2 \xrightarrow{\text{evolution}} S_3 \xleftarrow{\text{projection}}$$

Sheaf formalism

Input — Internal state — Output

State variables



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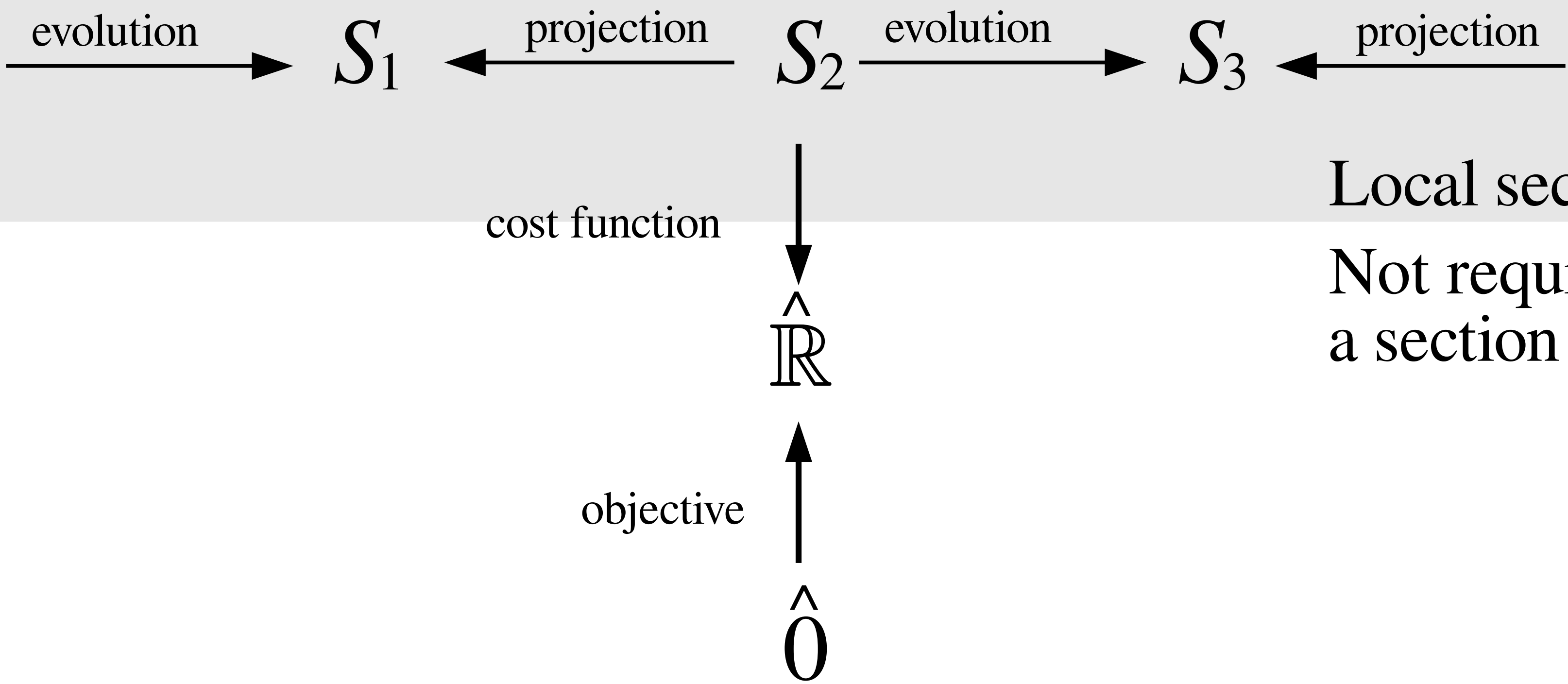
- Interpret the diagram of sheaves **as a sheaf itself**
- Global sections across the whole diagram are feasible timeseries when certain controls are applied

Optimally controlled systems

- Theorem: Optimal control is obtained by minimizing consistency radius on the diagram…



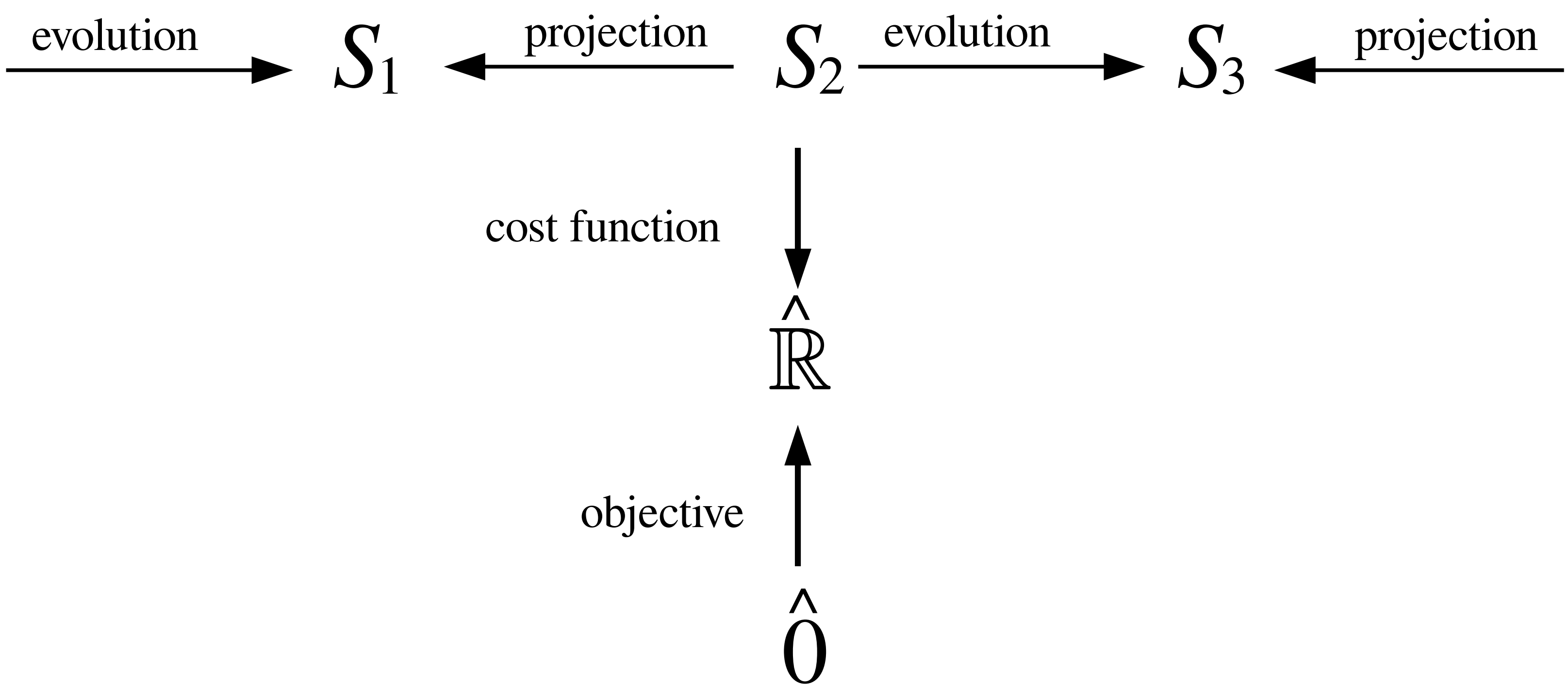
Local section here

Not required to be a section here

… subject to the constraint that the top row remains a section

Robustly controlled systems

- Allowing for inconsistency **throughout** allows for control that tolerates observation error



- Theorem: Consistency radius across the entire diagram is a lower bound* on control error

*See arxiv:2012.00120 for details

A practical topological filter

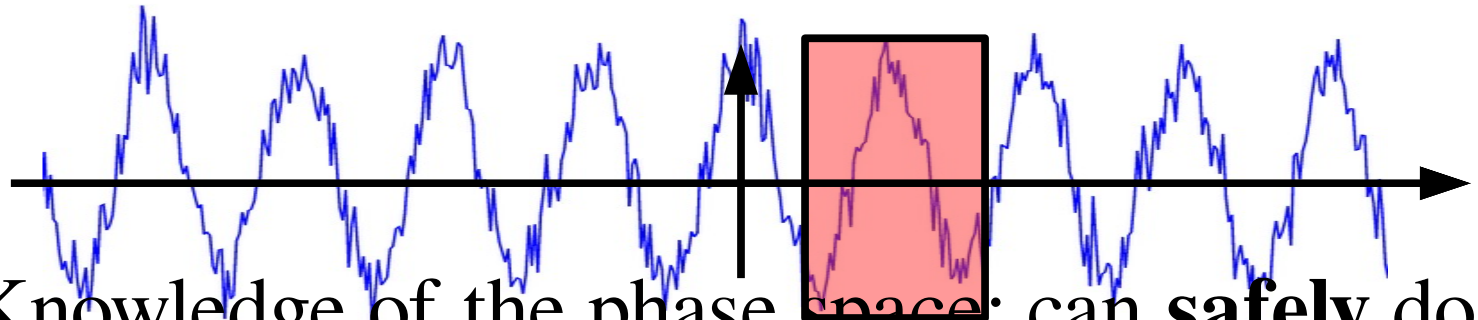
The *QuasiPeriodic Low Pass Filter*

(QPLPF)

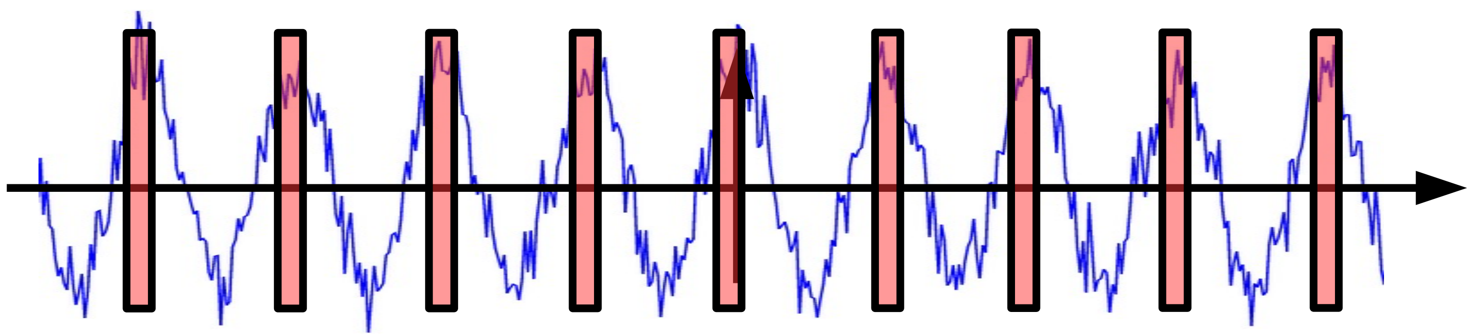
Michael Robinson

Circumventing bandwidth limits

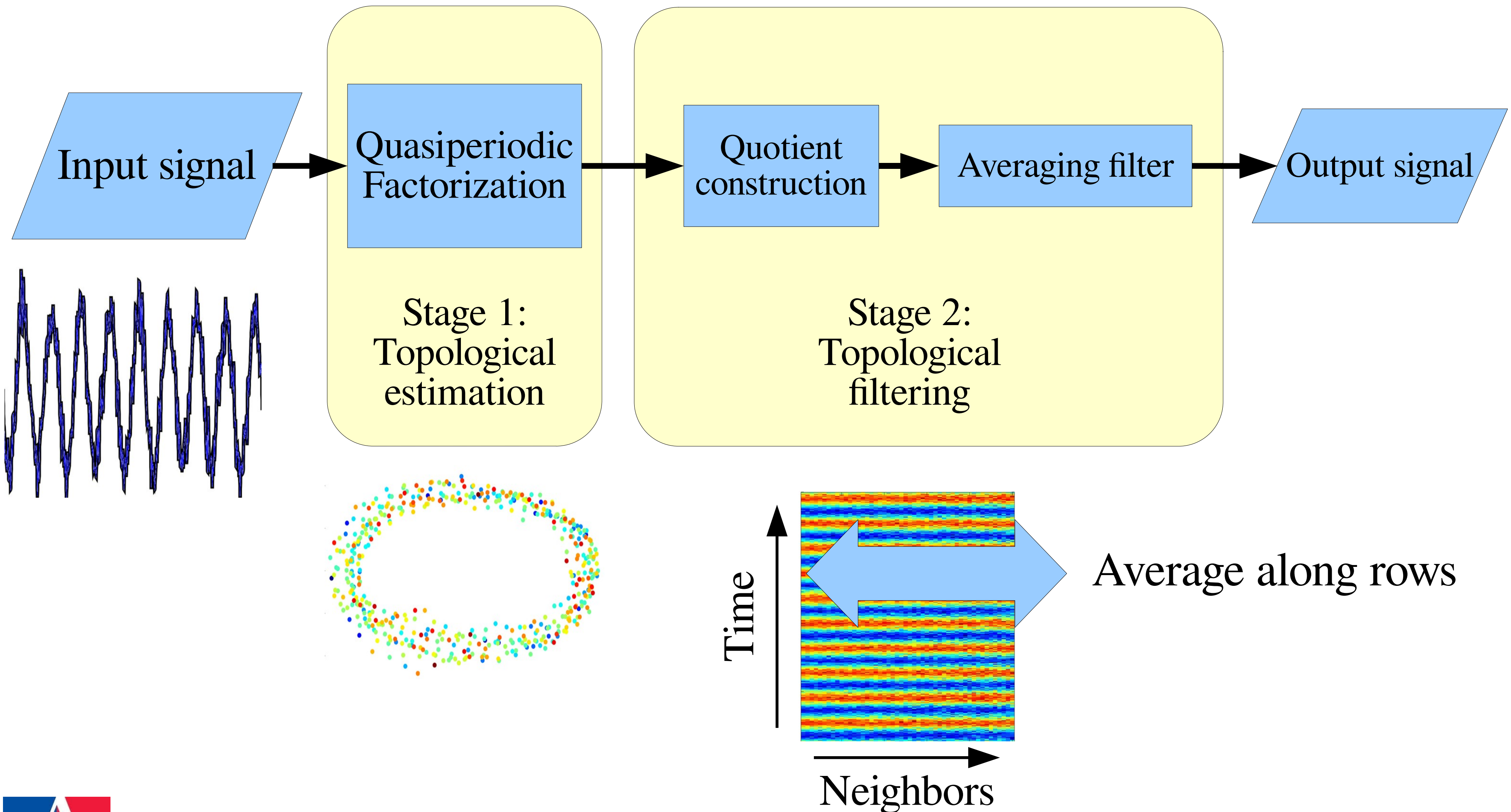
- Traditional: averaging in a connected window
  - Noise cancellation (Good)
  - Distortion to the signal (Bad)



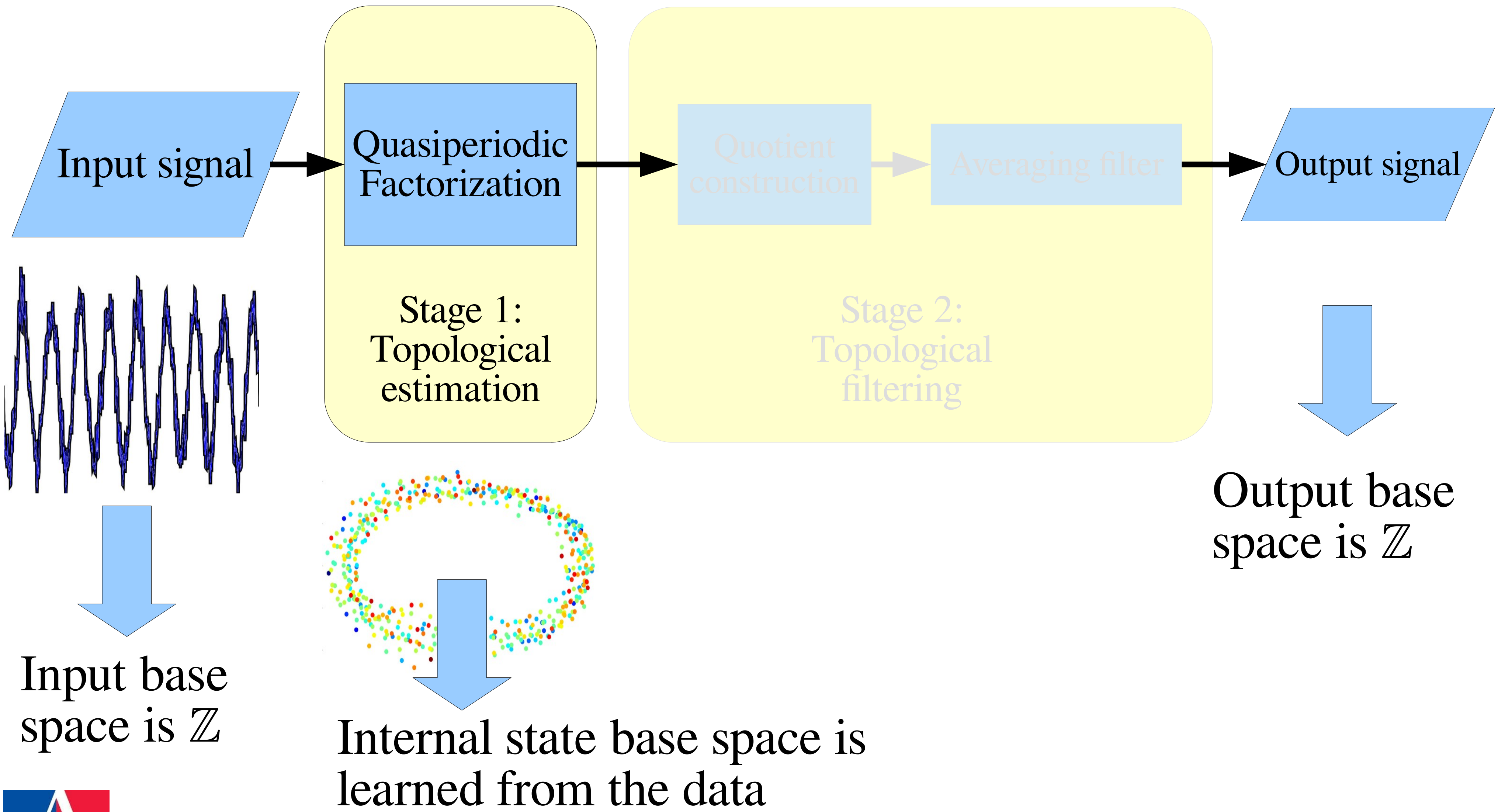
- Knowledge of the phase space: can **safely** do **more** averaging across the **entire** signal



QPLPF block diagram



How is this a topological filter?



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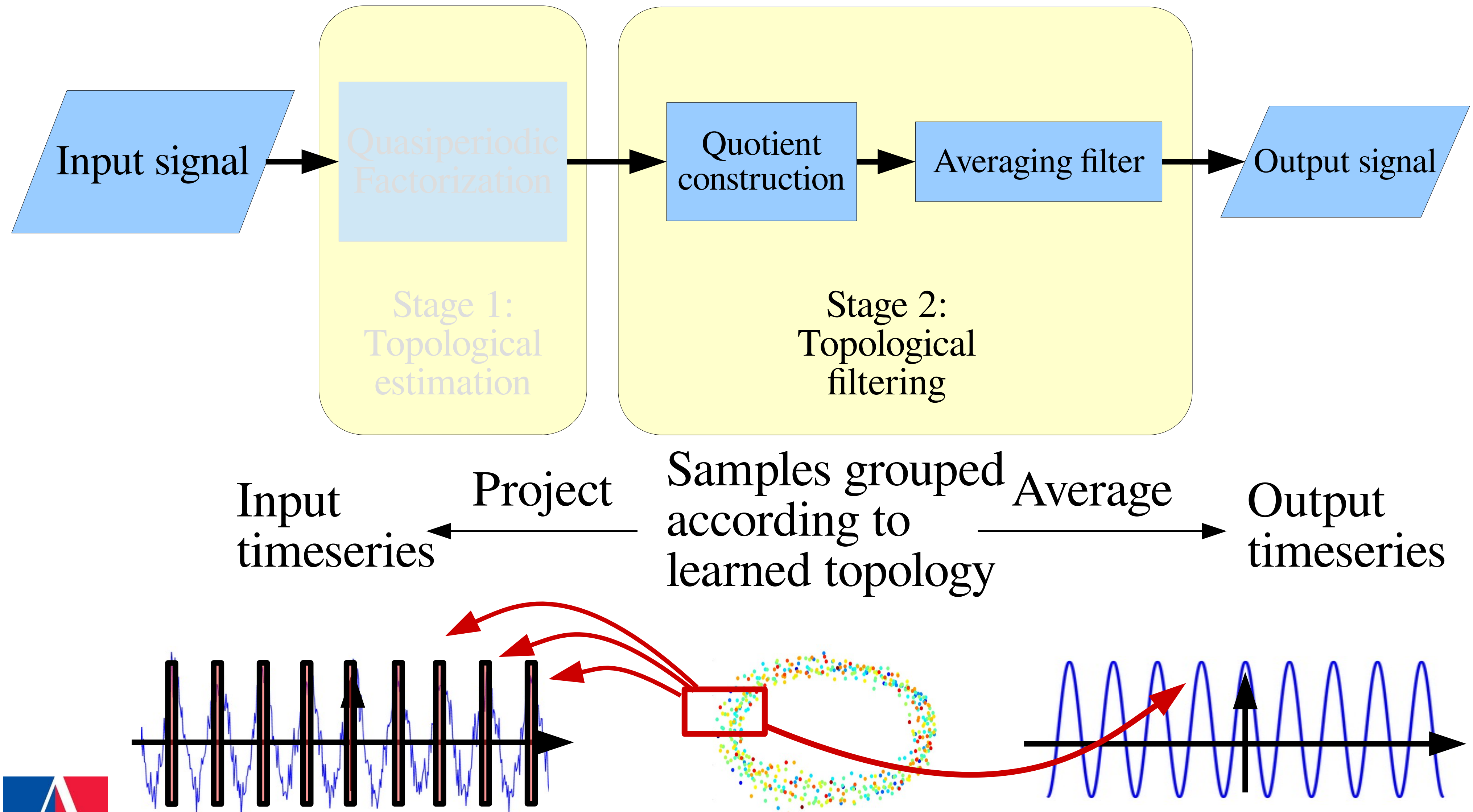
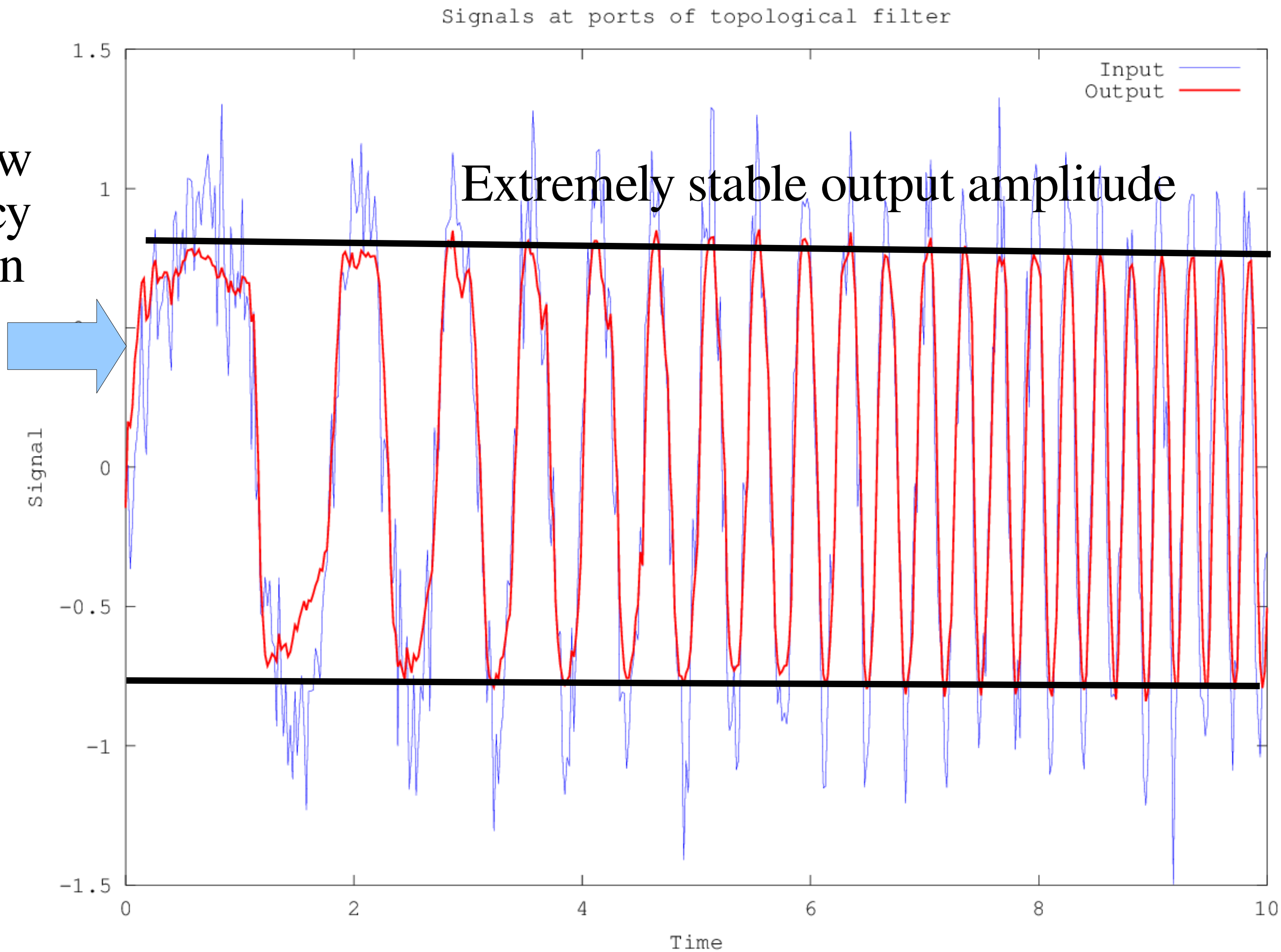


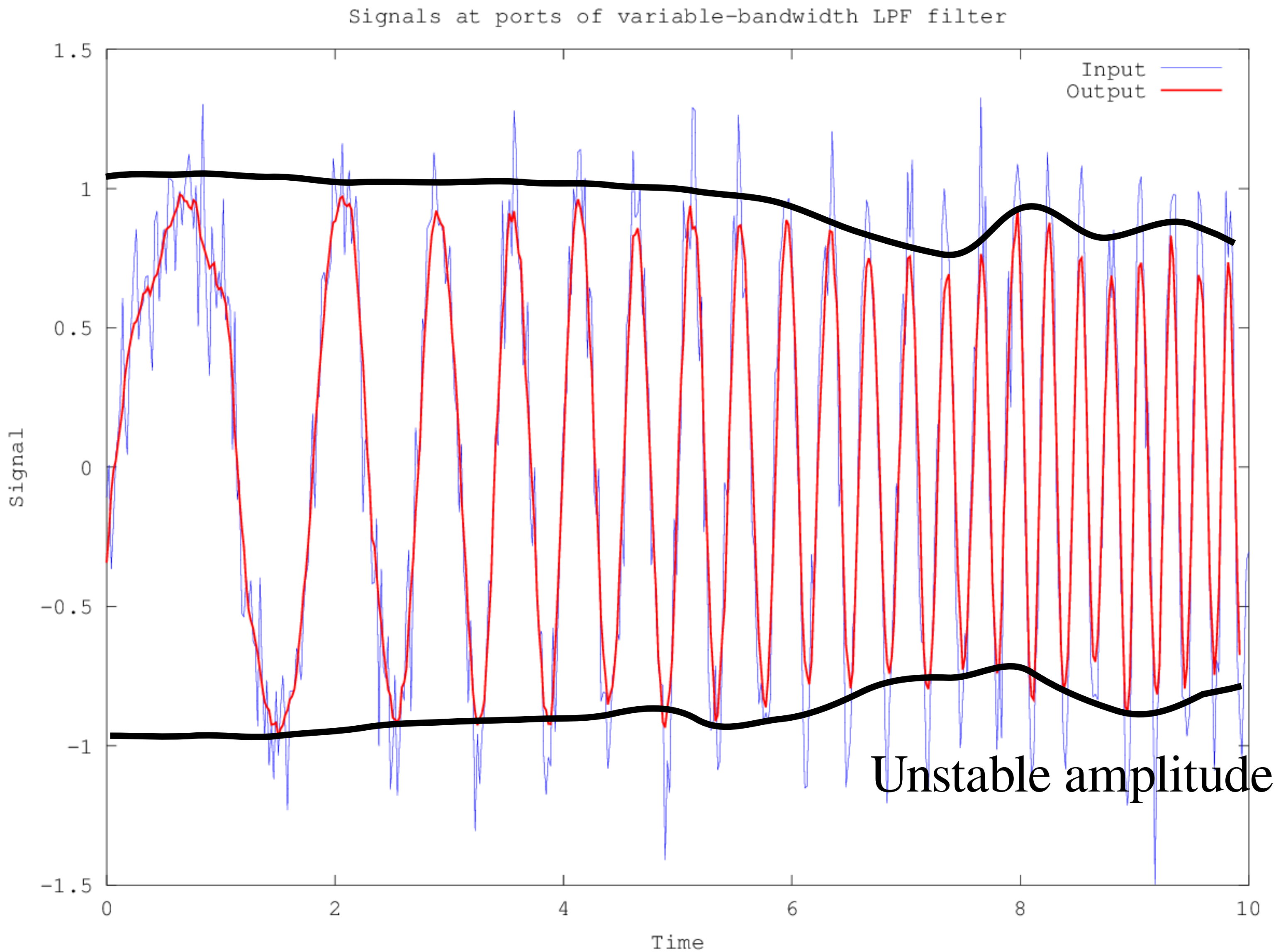
QPLPF results

Some low frequency distortion



Michael Robinson

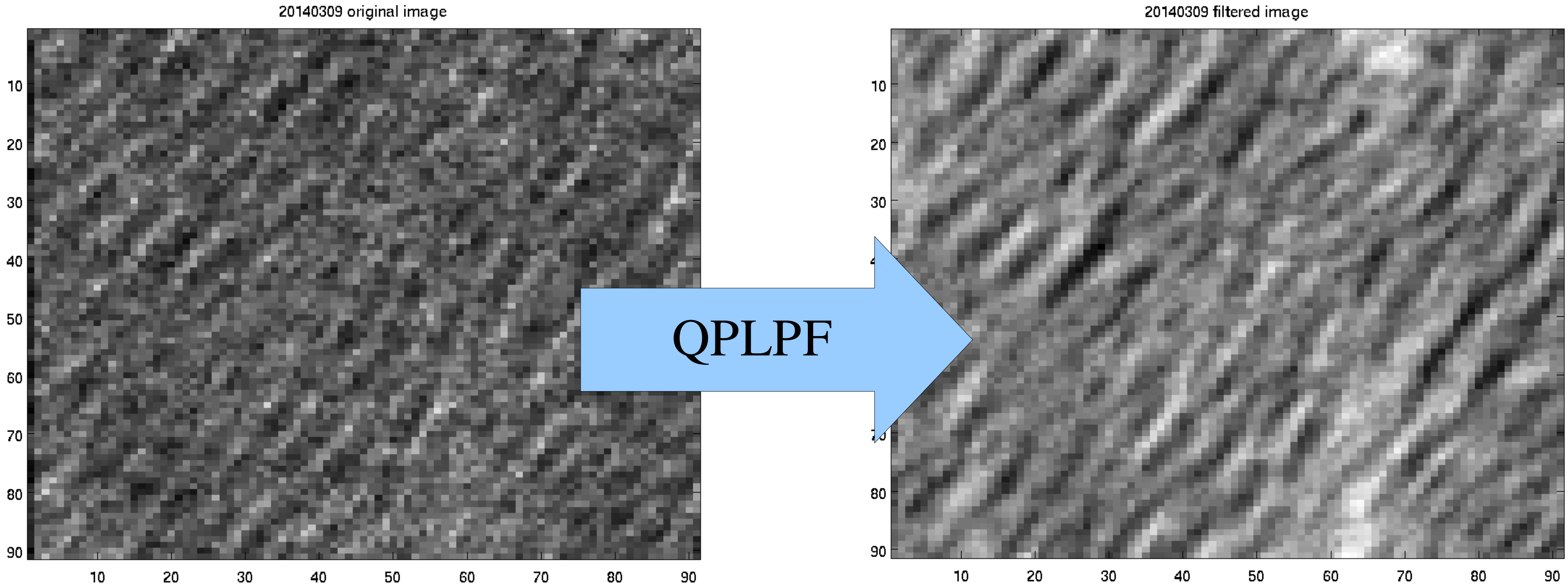
Compare: standard adaptive filter



Ocean radar image despeckling

After topological filtering:

- Speckle and contrast improved



Conclusions

- Sheaves encompass standard LTI filters in a way that corresponds to filter hardware
- Sheaf-based filters generalize LTI filters to nonlinear ones
  - The resulting filters can be tuned without too much effort
  - Interested? See the work of Georg Essl on "topological filtering"
- Lunch!